# Holomorphic Dynamic and Möbius Transformations

MTH3000 - Research Project

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# Contents

# V Semigroups



# VI Models

# Poincaré Model

# 1. Introduction

## 1.1 History

A brief history of the concepts that will be developed in this report. The two spaces we are interested in are known as the Poincaré disk and Poincaré half-space or upper half plane. Both of these are named after Henri Poincaré, but it was originally developed by Eugenio Beltrami who used it to show that hyperbolic geometry was equiconsistent with Euclidean geometry [8]. A natural question that arises in the study of models concerns the automorphisms of that space, and in the case of the upper half plane, these turn out to be the Möbius transformations. Möbius transformations were developed by August Ferdinand Möbius around 1827 during his study of analytical geometry. A group we will see later called the projective special linear group, which is very closely related to Mobius transformations, was developed by Evariste Galois in the 1830s, in the context of Lie Groups. Continuous one-parameter semigroups of holomorphic self-maps of the unit disc $\mathbb{D}$ in the complex plane $\mathbb{C}$ have been studied since the early 1900s, both for their intrinsic interest in complex analysis and for applications to areas such as differential equations [4].

## 1.2 Complex Analysis

To begin we state some standard definitions in complex analysis.

**Definition 1.2.1 — Cauchy Riemann Equations.**
Let $z = x + iy \in \mathbb{C}$ for $x, y \in \mathbb{R}$, and $f(x,y) = u(x,y) + iv(x,y)$. The Cauchy-Riemann Equations are given by,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

**Definition 1.2.2 — Holomorphic.**
A map $f : U \subset \mathbb{C} \mapsto \mathbb{C}$ is *holomorphic* if it satisfies the Cauchy Riemann Equations on $U$

**Definition 1.2.3 — Biholomorphism.**
A map $f : \mathbb{C} \mapsto \mathbb{C}$, is *biholomorphic* if $f$ and $f^{-1}$ are holomorphic.

The following two theorems are fundamental in complex analysis.

**Theorem 1.2.1 — Maximum Modulus Principle (2).**
Let $C$ be a simple closed contour in $\mathbb{C}$ and $f : C \mapsto \mathbb{C}$ be holomorphic, then for $z_0$ in the interior of $C$,

$$|f(z_0)| \leq \max_{z\in C} |f(z)|$$

with equality iff $f$ is constant on $C$.

We can use the maximum modulus to prove the following theorem about complex functions.

**Theorem 1.2.2 — Schwarz Lemma (4).**
Let $\varphi : \mathbb{D} \mapsto \mathbb{D}$ be holomorphic with $\varphi(0) = 0$. Then $|\varphi(z)| \leq |z|$ and $|\varphi'(0)| \leq 1$, with equality iff $\varphi$ is a rotation.

**Schwarz Lemma Proof**
Let $f(z) = \frac{\varphi(z)}{z}$ for $z \neq 0$ and $f(0) = \varphi'(0)$. The map $f : \mathbb{D} \mapsto \mathbb{C}$ is holomorphic. Let $0 < r < 1$ and $|z| \leq r$, by the maximum modulus principle,

$$|f(z)| \leq \max_{|z|\leq r} |f(z)| = \max_{|z|=r} \left|\frac{\varphi(z)}{z}\right| \leq \max_{|z|=r} \frac{1}{|z|} = \frac{1}{r}$$

Letting $r \to 1$ we obtain $|f(z)| = 1$, in particular $|\varphi(z)| \leq |z|$ and $|\varphi'(0)| = |f(0)| \leq 1$. If equality holds for some $z \in \mathbb{D}$, then $|f(z)|$ is constant on $\mathbb{D}$, so $\varphi(z) = e^{i\theta} z$ for some $\theta \in \mathbb{R}$. If $\varphi(z) = e^{i\theta} z$ then $|\varphi(z)| = |z|$ and $\varphi'(z) = e^{i\theta}$ so $|\varphi'(0)| = 1$ and hence the reverse direction holds.

The following function will be used frequently for its properties of being an automorphism, mapping a specified point to 0 and being its own inverse.

**Theorem 1.2.3 — Automorphisms of the unit disk.**
Let $a \in \mathbb{D}$ and $T_a : \bar{\mathbb{D}} \mapsto \mathbb{C}$ defined by,

$$T_a(z) = \frac{a - z}{1 - \bar{a}z}$$

Then $\|T_\alpha(z)\| \leq 1$ with equality iff $\|z\| = 1$. $T_a$ is also an automorphism of $\mathbb{D}$.

**Proof**
Let $z \in \partial\mathbb{D}$ and $\alpha \in \mathbb{D}$, then $\|z\| = z\bar{z} = 1$,

$$T_a(z) = \frac{a - z}{1 - \bar{a}z}$$

$$= \frac{a\bar{z}-1}{\bar{z}-\bar{a}} \qquad \left(\times \frac{\bar{z}}{\bar{z}}\right)$$

$$= \frac{(a\bar{z}-1)(z-a)}{\|z-a\|^2} \qquad \left(\times \frac{z-\alpha}{z-a}\right)$$

$$= \frac{(z-a)^2}{z\|z-a\|^2} \qquad \left(\times \frac{z}{z}\right)$$

Thus,

$$\|T_a(z)\| = 1$$

Since the only pole of $T_a$ is at $1/\bar{a}$ which is outside the disk, $T$ is holomorphic and the result follows from maximum modulus theorem.

$$\begin{aligned} T_a \circ T_a(z) &= \frac{a - T_a(z)}{1 - \bar{a}T_a(z)} \\ &= \frac{a - \frac{a-z}{1-\bar{a}z}}{1 - \bar{a}\frac{a-z}{1-\bar{a}z}} \\ &= \frac{a(1-\bar{a}z)-(a-z)}{1-\bar{a}z-\bar{a}(a-z)} \\ &= z \end{aligned}$$

$T_a$ is its own inverse thus $T_a$ is an automorphism of the unit disk.

The Schwarz Lemma is quite restrictive, requiring $f(0) = 0$, so we extend it to all holomorphic functions. To do this we use the Schwarz lemma along with the automorphism $T_a$,

**Theorem 1.2.4 — Schwarz Pick Lemma.**
Let $\varphi : \mathbb{D} \mapsto \mathbb{D}$ be holomorphic. Then for $u, v \in \mathbb{D}$,

$$|T_{\varphi(u)}(\varphi(v))| \leq |T_u(v)|$$

**Schwarz Pick Lemma Proof**
Let $F = T_{\varphi(u)} \circ \varphi \circ T_u$. Then,

$$F(0) = T_{\varphi(u)} \circ \varphi \circ T_u(0) = T_{\varphi(u)} \circ \varphi(u) = 0$$

Applying the Schwarz Lemma, $|F(z)| \leq |z|$. Let $z = T_u(v)$, then $|T_{\varphi(u)}(\varphi(v))| \leq |T_u(v)|$

# 2. Hyperbolic Metric

A general metric tensor on the complex plane is given by,

$$ds^2 = \lambda^2(z,\bar{z})\,\mathrm{d}z\,\mathrm{d}\bar{z}$$

the length of a curve $\gamma$ in the complex plane is given by,

$$\ell(\gamma) = \int_\gamma \mathrm{d}s = \int_\gamma \lambda(z)|\,\mathrm{d}z\,|$$

and a metric for our space is given by

$$d(u,v) = \inf_\gamma \ell(\gamma)$$

where $\gamma$ is any curve in the space from $u$ to $v$.

We will construct our 'density function' $\lambda$ such that $T_a$ is an isometry for the metric $d$ on the unit disk $\mathbb{D}$, namely $d(u,v) = d(T_a(u), T_a(v))$ for any $u, v$.

**Proposition 2.0.1** The density function $\lambda$ is given by

$$\lambda(z) = \frac{2}{1 - \|z\|^2}$$

**Proof**
Let $u, v \in \mathbb{D}$ and $T_a$ an automorphism of the disk. Then,

$$\begin{aligned} d(u,v) &= \inf_\gamma \int_\gamma \lambda(z)|\,\mathrm{d}z\,| \\ d(T_a(u), T_a(v)) &= \inf_{\gamma'} \int_{\gamma'} \lambda(z)|\,\mathrm{d}z\,| \\ &= \inf_\gamma \int_\gamma \lambda(T_a(z))|T_a'(z)||\,\mathrm{d}z\,| \end{aligned}$$

Thus a sufficient condition for $T_a$ to be an isometry is $\lambda(T_a(z))|T_a'(z)| = \lambda(z)$.

$$|T_a'(z)| = \frac{|1 - \bar{a}z + \bar{a}(z-a)|}{|1 - \bar{a}z|^2}$$

$$
\begin{aligned}
&= \frac{1-|a|^2}{|1-\bar{a}z|^2} \\
\lambda(T_a(z)) &= \frac{2}{1-|T_a(z)|^2} \\
&= \frac{2}{1-|\frac{z-a}{1-\bar{a}z}|^2} \\
&= \frac{2|1-\bar{a}z|^2}{|1-\bar{a}z|^2-|z-a|^2} \\
&= \frac{2|1-\bar{a}z|^2}{1-2\operatorname{Re}(\bar{a}z)+|\bar{a}z|^2-(|z|^2-2\operatorname{Re}(a\bar{z})+|a|^2)} \\
&= \frac{2|1-\bar{a}z|^2}{(1-|a|^2)(1-|z|^2)}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\lambda(T_a(z))|T_a'(z)| &= \frac{2|1-\bar{a}z|^2}{(1-|a|^2)(1-|z|^2)}\frac{1-|a|^2}{|1-\bar{a}z|^2} \\
&= \frac{2}{1-|z|^2} \\
&= \lambda(z)
\end{aligned}
$$

The **Poincaré Disk** is the unit disk in the complex plane along with the Riemann metric given above. $\mathbb{D}$ refers to the Poincaré disk unless stated otherwise.

**Example 2.1 — Distance to $0$.**
We can find an explicit formula for the distance to the origin using the above integral. Let $u \in \mathbb{D}$, then $\gamma(t) = ut$ for $0 \leq t \leq 1$

$$
\begin{aligned}
d(0,u) &= \int_\gamma \frac{2}{1-|z|^2}|\,\mathrm{d}z\,| \\
&= \int_0^1 \frac{2}{1-|u|^2t^2}|u|\,\mathrm{d}t \\
&= |u|\int_0^1 \frac{1}{1+|u|t}+\frac{1}{1-|u|t}\,\mathrm{d}t \\
&= |u|\left(\frac{1}{|u|}\ln(1+|u|t)-\frac{1}{|u|}\ln(1-|u|t)\right)_0^1 \\
&= \ln(1+|u|)-\ln(1-|u|) \\
&= \ln\left(\frac{1+|u|}{1-|u|}\right)
\end{aligned}
$$

# 3. Geometry

## 3.1 Geodesics

Let $u, v \in \mathbb{D}$, the geodesic between $u$ and $v$ is the shortest path joining them.

**Definition 3.1.1** Let $u, v \in \mathbb{D}$ and $\gamma$ a curve in $\mathbb{D}$ passing through $u, v$. $\gamma$ is a geodesic if

$$d(u,v) = \int_\gamma \lambda(z)\,\mathrm{d}z$$

Before we find an explicit formula for a geodesic between two points, we need some properties of geodesics. Since our density function, $\lambda$ only depends on $|z|$, the geodesic between the origin and another point is a straight line.

Let $u, v$ lie on the interior of $\mathbb{D}$. The automorphism $\varphi(z) = \frac{z-u}{1-\bar{u}z}$ maps $u \mapsto 0$. As $\varphi$ is an isometry for the unit disk, it maps the geodesic between $u$ and $v$ goes to the straight line between $0$ and $\varphi(v)$. Thus if we extend the geodesic towards the boundary, the straight line intersects the boundary at right angles, and since the automorphisms are conformal maps [10], the geodesic between $u$ and $v$ intersects the boundary at right angles. Moreover Möbius transformations take circles to circles, and a straight line can be considered as a circle through infinity on the Riemann sphere, thus the geodesic between $u$ and $v$ is an arc of a circle. To find the geodesics we make use of the following proposition,

**Proposition 3.1.1** In the Euclidean plane $\mathbb{R}^2$, the circle $x^2+y^2+ax+by+1=0$ intersects the circle $x^2+y^2=1$ at right angles.

**Proof**
Let $C_1$ denote the circle $x^2+y^2+ax+by+1=0$ and $C_2$ denote $x^2+y^2=1$. Substituting $C_2$ into $C_1$,

$$ax+by+2=0$$

Differentiating $C_1$ w.r.t $x$,

$$2x+a+(2y+b)y'=0$$

Thus a tangent vector is given by

$$(1,y') = \left(1, -\frac{2x+a}{2y+b}\right)$$

Similarly for $C_1$,

$$2x+2yy'=0 \Rightarrow (1,y')=\left(1,\frac{x}{y}\right)$$

Thus,

$$\begin{aligned}
C_1'\cdot C_2' &= \left(1,-\frac{2x+a}{2y+b}\right)\cdot\left(1,\frac{x}{y}\right)\\
&= 1+\frac{2x+a}{2y+b}\left(1,\frac{x}{y}\right)\\
&= \frac{y(2y+b)+x(2x+a)}{(2y+b)y}\\
&= \frac{2+ax+by}{(2y+b)y}
\end{aligned}$$

Since $ax+by+2=0$ at the intersection of $C_1$ and $C_2$,

$$C_1'\cdot C_2'=0$$

Therefore $C_1$ intersects $C_2$ at right angles.

**Example 3.1 — Formula for geodesics.**
Let $v=v_0+iv_1, w=w_0+iw_1\in\partial\mathbb{D}$. The geodesic passing through $v$ and $w$ is given by $x^2+y^2+ax+by+1=0$ for appropriate $a$ and $b$. Substituting $v$ and $w$,

$$\begin{aligned}
0 &= v_0^2+v_1^2+av_0+bv_1+1\\
0 &= w_0^2+w_1^2+aw_0+bw_1+1\\
\begin{pmatrix} v_0 & v_1\\ w_0 & w_1\end{pmatrix}\begin{pmatrix} a\\ b\end{pmatrix} &= \begin{pmatrix} -1-v_0^2-v_1^2\\ -1-w_0^2-w_1^2\end{pmatrix}\\
\begin{pmatrix} a\\ b\end{pmatrix} &= \frac{-1}{v_0w_1-v_1w_0}\begin{pmatrix} w_1 & -v_1\\ -w_0 & v_0\end{pmatrix}\begin{pmatrix} 1+|v|^2\\ 1+|w|^2\end{pmatrix}
\end{aligned}$$

Therefore,

$$\begin{aligned}
a &= \frac{v_1(1+|w|^2)-w_1(1+|v|^2)}{v_0w_1-v_1w_0}\\
b &= \frac{w_0(1+|v|^2)-v_0(1+|w|^2)}{v_0w_1-v_1w_0}
\end{aligned}$$

Substituting these into the formula for the circle, $x^2+y^2+ax+by+1=0$, we obtain an explicit expression for the geodesic between $v$ and $w$

## 3.2 Triangles

Now we find a formula for the area of a triangle in the disk. Using the results from the previous section, the Riemann metric for the unit disk is given by

$$\mathrm{d}s^2=\frac{4}{(1-|z|^2)^2}|\,\mathrm{d}z\,|^2$$

Writing $\mathrm{d}z = \mathrm{d}x + \mathrm{d}y\,i$, we find the first fundamental form for $\mathrm{d}s^2$ $\left(\mathrm{d}s^2 = E\,\mathrm{d}x^2 + 2F\,\mathrm{d}x\,\mathrm{d}y + G\,\mathrm{d}y^2\right)$ has coefficients,

$$\begin{aligned} E &= \frac{4}{(1-|z|^2)^2} \\ F &= 0 \\ G &= \frac{4}{(1-|z|^2)^2} \end{aligned}$$

Thus we can compute the curvature of $\mathbb{D}$ using the following formula [6]

**Theorem 3.2.1** For a parametrisation with $F = 0$, the Gaussian curvature $K$ is given by,

$$K = \frac{-1}{2\sqrt{EG}}\left(\frac{\partial}{\partial x}\frac{G_x}{\sqrt{EG}} + \frac{\partial}{\partial y}\frac{E_y}{\sqrt{EG}}\right)$$

**Example 3.2** Calculating $K$ for the Poincaré disk.

$$\sqrt{EG} = E$$

$$\begin{aligned} G_x &= \frac{\partial}{\partial x}\frac{4}{1-x^2-y^2} \\ &= \frac{16x}{(1-x^2-y^2)^3} \\ \frac{\partial}{\partial x}\frac{G_x}{\sqrt{EG}} &= \frac{\partial}{\partial x}\frac{16x}{1-x^2-y^2} \\ &= \frac{4(1-x^2-y^2)+8x^2}{(1-x^2-y^2)^2} \\ &= \frac{4(1+x^2-y^2)}{(1-x^2-y^2)^2} \end{aligned}$$

$$\begin{aligned} E_y &= \frac{\partial}{\partial y}\frac{4}{1-x^2-y^2} \\ &= \frac{16y}{(1-x^2-y^2)^3} \\ \frac{\partial}{\partial y}\frac{E_y}{\sqrt{EG}} &= \frac{\partial}{\partial y}\frac{16y}{1-x^2-y^2} \\ &= \frac{4(1-x^2-y^2)+86^2}{(1-x^2-y^2)^2} \\ &= \frac{4(1-x^2+y^2)}{(1-x^2-y^2)^2} \end{aligned}$$

$$\begin{aligned} K &= \frac{-1}{2\sqrt{EG}}\left(\frac{\partial}{\partial x}\frac{G_x}{\sqrt{EG}} + \frac{\partial}{\partial y}\frac{E_y}{\sqrt{EG}}\right) \\ &= \frac{-(1-x^2-y^2)^2}{8}\left(\frac{4(1+x^2-y^2)}{(1-x^2-y^2)^2} + \frac{4(1-x^2+y^2)}{(1-x^2-y^2)^2}\right) \\ &= \frac{-(1-x^2-y^2)^2}{8}\frac{8}{(1-x^2-y^2)^2} \\ &= -1 \end{aligned}$$

Therefore $\mathbb{D}$ has Gaussian curvature $-1$. The Poincaré Disk is referred to as a model for the hyperbolic plane since they both have constant negative curvature.

For a general triangle in the Poincaré disk, we can calculate its area using Gauss-Bonnet [5]

**Theorem 3.2.2 — Gauss Bonnet.**
Suppose $M$ is a compact two dimensional Riemannian manifold with boundary $\partial M$. Let $K$ be the Gaussian curvature of $M$ and $k_g$ be the geodesic curvature of $\partial M$. Then,

$$\int_M K\,\mathrm{d}A + \int_{\partial M} k_g\,\mathrm{d}s = 2\pi\chi(M)$$

where d$A$ is the area element, d$s$ is the line element and $\chi(M)$ is the Euler characteristic of $M$. Reformulated for a geodesic triangle $T$ (the sides of the triangle lie of geodesics),

$$\int_T K = \alpha + \beta + \gamma - \pi$$

where $\alpha, \beta, \gamma$ are the interior angles of $T$.

**Example 3.3 — Area in $\mathbb{D}$.**
We found previously that $K = -1$, therefore

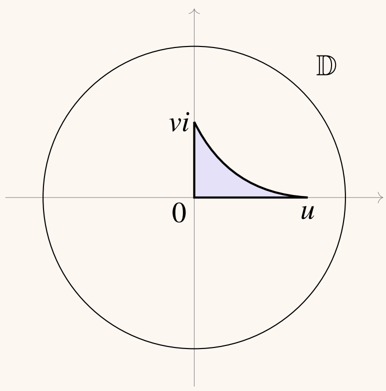
$$\int_T = A = \pi - (\alpha + \beta + \gamma)$$

For right angled triangles we have a nice relationship between the side lengths not dissimilar from the Pythagorean Theorem for Euclidean right angled triangles.

**Theorem 3.2.3 — Hyperbolic Pythagoras' Theorem.**
Let $\triangle(ABC)$ be a right angled triangle in $\mathbb{D}$ with one vertex at the origin, side lengths $a$, $b$ and hypotenuse $c$. Then,

$$\cosh a \cosh b = \cosh c$$

**Hyperbolic Pythagoras' Theorem Proof**
Without loss of generality, we assume $B = u$ lies on the Re axis and $C = vi$ on the Im axis, since rotations are isometries of the unit disk.




The geodesic from 0 to $u$ is the straight line lying on the real axis, thus the integral in $d_{\mathbb{D}}$

becomes a standard single variable integral,

$$\begin{aligned} a &= \int_0^u \frac{2}{1-t^2}\,\mathrm{d}t \\ &= \int_0^u \frac{1}{1+t} + \frac{1}{1-t}\,\mathrm{d}t \\ &= \ln(1+t) - \ln(1-t)\big|_{t=0}^{u} \\ &= \ln\left(\frac{1+u}{1-u}\right) \end{aligned}$$

Similarly for $v$

$$b = \ln\left(\frac{1+v}{1-v}\right)$$

Since the line $BC$ doesn't lie on a straight line through the origin, we can't use the same procedure. Instead take the automorphism of the unit disk,

$$\varphi(z) = \frac{z - vi}{1 + viz}$$

This maps $v \mapsto 0$ and $u \mapsto \frac{u-vi}{1+uiv}$. Then we take the integral over $\gamma(t) = \varphi(u)t$ for $t \in [0,1]$.

$$\begin{aligned} c &= \int_0^1 \frac{2|\varphi(u)|}{1-|\varphi(u)|^2t^2}\,\mathrm{d}t \\ &= |\varphi(u)| \int_0^1 \frac{1}{1+|\varphi(u)t} + \frac{1}{1-|\varphi(u)t}\,\mathrm{d}t \\ &= \ln\left(\frac{1+|\varphi(u)|}{1-|\varphi(u)|}\right) \end{aligned}$$

Now we build up to the LHS of the equality,

$$\begin{aligned} \cosh x &= \frac{e^x + e^{-x}}{2} \\ \cosh \ln x &= \frac{e^{\ln x} + e^{-\ln x}}{2} \\ &= \frac{x + \frac{1}{x}}{2} \\ \cos \ln \frac{1+u}{1-u} &= \frac{\frac{1+u}{1-u} + \frac{1-u}{1+u}}{2} \\ &= \frac{(1+u)^2 + (1-u)^2}{2(1-u)^2} \\ &= \frac{1+u^2}{1-u^2} \end{aligned}$$

Therefore,

$$\cosh a \cosh b = \frac{1+u^2}{1-u^2}\frac{1+v^2}{1-v^2}$$

Switching our attention to the RHS,

$$\begin{aligned}
\cosh c &= \frac{1+|\varphi(u)|^2}{1-|\varphi(u)|^2} \\
&= \frac{1+\left|\frac{u-vi}{1+uiv}\right|^2}{1-\left|\frac{u-vi}{1+uiv}\right|^2} \\
&= \frac{|1+uvi|^2+|u-vi|^2}{|1+uvi|^2-|u-vi|^2} \\
&= \frac{1+u^2v^2+u^2+v^2}{1+u^2v^2-u^2-v^2} \\
&= \frac{(1+u^2)(1+v^2)}{(1-u^2)(1-v^2)}
\end{aligned}$$

Therefore,

$$\cosh a \cosh b = \cosh c$$

# II

# Upper Half Plane

# 4. Hyperbolic Metric

Similarly to the Poincaré Disk, we will find the Riemann metric for the upper half plane $\{z \in \mathbb{C} \mid \operatorname{Im}(z) > 0\}$. To avoid confusion we will denote the density and Riemann metric on $\mathbb{D}$ as $\lambda_{\mathbb{D}}$ and $d_{\mathbb{D}}$ respectively and the corresponding functions on the upper half plane as $\lambda_{\mathbb{H}}$ and $d_{\mathbb{H}}$.

**Proposition 4.0.1** The density function $\lambda_{\mathbb{H}}$ for the upper half plane preserving the metric in Poincaré disk is given by,

$$\lambda_{\mathbb{H}}(z) = \frac{1}{\operatorname{Im}(z)}$$

**Proof**
Let $\varphi : \mathbb{D} \mapsto \mathbb{H}$ be defined by $\varphi(z) = \frac{z+i}{1+iz}$. Then similarly to the Poincaré disk, we can use change of variables to move from $\mathbb{H}$ to $\mathbb{D}$,

$$\begin{aligned}
\lambda_{\mathbb{D}}(\varphi(z)) &= \frac{1}{1-|\varphi(z)|^2} \\
&= \frac{1|1+iz|^2}{|1+iz|^2-|z+i|^2} \\
&= \frac{|1+iz|^2}{2\operatorname{Im}(z)} \\
|\varphi'(z)| &= \frac{2}{|1+iz|^2} \\
\lambda_{\mathbb{H}}(z) &= |\varphi'(z)|\lambda_{\mathbb{D}}(\varphi(z)) \\
&= \frac{2}{|1+iz|^2}\frac{|1+iz|^2}{2\operatorname{Im}(z)} \\
&= \frac{1}{\operatorname{Im}(z)}
\end{aligned}$$

When we studied the Poincaré disk, we started with automorphisms of the unit disk, and found a Riemann metric with those automorphisms as the isometries for the metric. Now we have a Riemann metric on the upper half plane, and we want to find the automorphisms of $\mathbb{H}$ which are isometries for $d_{\mathbb{H}}$.

**Definition 4.0.1** Let $a,b,c,d \in \mathbb{C}$ satisfying $ad-bc \neq 0$ then

$$A(z) = \frac{az+b}{cz+d}$$

is a Möbius transformation.

We can identify the coefficients $a, b, c, d$ with the entries of a 2x2 matrix, namely

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \frac{az+d}{cz+d}$$

Suppose we have a Möbius Transformation $\varphi$ with coefficients $a, b, c, d \in \mathbb{R}$.

**Proposition 4.0.2** The map $f$ defined by identifying the matrix entries with a Möbius transformation is a homomorphism from $\mathrm{SL}(2)$ to the group of Möbius transformations under composition.

**Proof**
We need to check that $f(xy) = f(x)f(y)$, or equivalently $f(xy)(z) = f(x) \circ f(y)(z)$ for all $z \in \mathbb{C}$. Let $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, y = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R})$.

$$\begin{aligned}
f(xy)(z) &= f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix}\right)(z) \\
&= f\begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}(z) \\
&= \frac{(ae+bg)z+af+bh}{(ce+dg)z+cf+dh} \\
f(x) \circ f(y)(z) &= f\begin{pmatrix} a & b \\ c & d \end{pmatrix} \circ f\begin{pmatrix} e & f \\ g & h \end{pmatrix}(z) \\
&= f\begin{pmatrix} a & b \\ c & d \end{pmatrix}\left(\frac{ez+f}{gz+h}\right) \\
&= \frac{a\frac{ez+f}{gz+h}+b}{c\frac{ez+f}{gz+h}+d} \\
&= \frac{a(ez+f)+b(gz+h)}{c(ez+f)+d(gz+h)} \\
&= f(xy)(z)
\end{aligned}$$

This homomorphism is surjective with kernel consisting of $\{I, -I\}$. Thus we call the quotient $\mathrm{SL}(2, \mathbb{R})/\{I, -I\}$ the projective special linear group, $\mathrm{PSL}(2, \mathbb{R})$ which is isomorphic to the group of Möbius transformations.

From this point onwards, it is assumed that a Möbius transformation $\varphi$ has real coefficients $a, b, c, d \in \mathbb{R}$ such that $ad - bc = 1$ unless stated otherwise.

**Proposition 4.0.3** The Möbius transformations are automorphisms of $\mathbb{H}$ and isometries for $d_{\mathbb{H}}$.

**Proof**
$\varphi$ maps the real axis to itself, so it suffices to check that a point on the interior of $\mathbb{H}$ is mapped back into the interior of $\mathbb{H}$.

$$\begin{aligned}\varphi(i) &= \frac{ai+b}{ci+d} \\ &= \frac{bd+ac+i}{c^2+d^2}\end{aligned}$$

Therefore $\varphi(i)$ is in the interior of $\mathbb{H}$ and $\varphi$ is an automorphism of $\mathbb{H}$.
To show $\varphi$ is an isometry for $\mathbb{H}$, we have the same sufficient condition as the unit disk,

$$\lambda_{\mathbb{H}}(z) = |\varphi'(z)|\lambda_{\mathbb{H}}(\varphi(z))$$

for any $z \in \mathbb{H}$ then $\varphi$ is an isometry. Let $z = x+iy \in \mathbb{C}$

$$\begin{aligned}\varphi(z) &= \frac{a(x+iy)+b}{c(x+iy)+d} \\ &= \frac{(ax+b+iay)(cx+d-icy)}{(cx+d)^2+y^2} \\ \operatorname{Im}(\varphi(z)) &= \frac{ay(cx+d)-cy(ax+b)}{|cz+d|^2} \\ &= \frac{xy(ac-ac)+y(ad-bc)}{|cz+d|^2} \\ &= \frac{\operatorname{Im}(z)}{|cz+d|^2} \\ |\varphi'(z)| &= \frac{a(cz+d)-c(az+b)}{|cz+d|^2} \\ &= \frac{1}{|cz+d|^2} \\ |\varphi'(z)|\lambda_{\mathbb{H}}(\varphi(z)) &= |\varphi'(z)|\frac{1}{\operatorname{Im}(\varphi(z))} \\ &= \frac{1}{|cz+d|^2}\frac{|cz+d|^2}{\operatorname{Im}(z)} \\ &= \lambda_{\mathbb{H}}(z)\end{aligned}$$

Therefore $\varphi$ is an isometry of $\mathbb{H}$.

Möbius Transformations will be studied in more detail in the next chapter.

# 5. Geometry

## 5.1 Geodesics

As with the Poincaré Disk, we will describe the geodesics and triangles of the upper half plane. Let $u, v \in \mathbb{D}$, the geodesic between $u$ and $v$ is the shortest path joining them.

**Definition 5.1.1** Let $u, v \in \mathbb{D}$ and $\gamma$ a curve in $\mathbb{H}$ passing through $u, v$. $\gamma$ is a geodesic if

$$d_{\mathbb{H}}(u,v) = \int_{\gamma} \lambda_{\mathbb{H}}(z)\,\mathrm{d}z$$

The discussion at the beginning of section 6.1 applies here, we want circles that intersect the real axis at right angles.

**Proposition 5.1.1** In the Euclidean plane $\mathbb{R}^2$, the circle $(x-a)^2 + y^2 = r^2$ intersects the x axis at right angles.

Thus to find the geodesic we solve the linear system with $u = u_0 + u_1 i$ and $v = v_0 + v_1 i$ lying on that line for $a, r$,

$$a = \frac{|u|^2 - |v|^2}{2(u_0 + v_0)}$$
$$r^2 = (u_0 - a)^2 + u_1^2$$

## 5.2 Triangles

Following the same methodology as in Section 6.2, we find the upper half plane has constant negative curvature $-1$. Thus $A = \pi - (\alpha + \beta + \gamma)$

# III Möbius Transformations

# 6. Classification

To motivate our classification, we examine the fixed points of a Möbius transformation. Let $\varphi$ be a Möbius transformation with coefficients given by $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2,\mathbb{R})$. A fixed point of $\varphi$ is a point $z \in \mathbb{C}$ such that $\varphi(z) = z$, thus

$$\begin{aligned} z &= \frac{az+b}{cz+d} \\ z(cz+d) &= az+b \\ 0 &= cz^2 + (d-a)z - b \end{aligned}$$

$$\begin{aligned} \Delta &= (a-d)^2 + 4bc \\ &= (a-d)^2 + 4(ad-1) \\ &= (a+d)^2 - 4 \end{aligned}$$

Thus since $a,d$ are real, we get two real fixed points when $(a+d)^2 > 4$, one fixed point when $(a+d)^2 = 4$ and two complex fixed points when $(a+d) \in [0,4)$.

A useful property of the trace is given by the following,

**Proposition 6.0.1** Let $D,B \in \mathrm{SL}(2,\mathbb{R})$, then

$$\operatorname{tr} DBD^{-1} = \operatorname{tr} B$$

**Proof**
Let $A \in \mathrm{SL}(2,\mathbb{R})$ and $B \in \mathrm{SL}(2,\mathbb{R})$, then the $\operatorname{tr} AB$ is the sum of the eigenvalues of $AB$, which is given by the characteristic polynomial,

$$\begin{aligned} \det(BA - \lambda I) &= \det\left(B^{-1}(BA - \lambda I)B\right) \\ &= \det(AB - \lambda I) \\ &= 0 \end{aligned}$$

Thus $AB$ and $BA$ have the same eigenvalues and therefore $\operatorname{tr} AB = \operatorname{tr} BA$.

Let $D,B \in \mathrm{SL}(2,\mathbb{R})$, then $\operatorname{tr}\left(DBD^{-1}\right) = \operatorname{tr}\left((DB)D^{-1}\right) = \operatorname{tr}\left(D^{-1}(DB)\right) = \operatorname{tr} B$

## 6.1 Parabolic

**Definition 6.1.1** A Möbius transformation $\varphi$ is Parabolic if $\varphi$ has one fixed point.

Equivalently, $(a+d)^2 = 4$. We note that since the coefficients are real, the fixed point lies on the boundary of $\mathbb{H}$. An example of a parabolic transformation is the translation $z \to z+2$ in $\mathbb{H}$. Geometrically parabolic transformations are translations in the upper half complex plane, more specifically they're translation around a horoball at the fixed point.

**Definition 6.1.2** Let $p$ be a point on the boundary of $\mathbb{H}$, Let $C_p$ be a Euclidean circle tangent to the boundary at $p$. Then the region bound by $C_p$ is a horoball at $p$, denoted $H_p$. Note if $p$ is the point at $\infty$ then $C_\infty$ is a horizontal line and $H_\infty$ is the Euclidean half plane above $C_\infty$

**Proposition 6.1.1** Let $A(z) = z + a$, for $a \in \mathbb{R}$. Then $A$ fixes $H_\infty$.

This follows from the definition of $H_\infty$, now we generalise this proposition to arbitrary hyperbolic transformations.

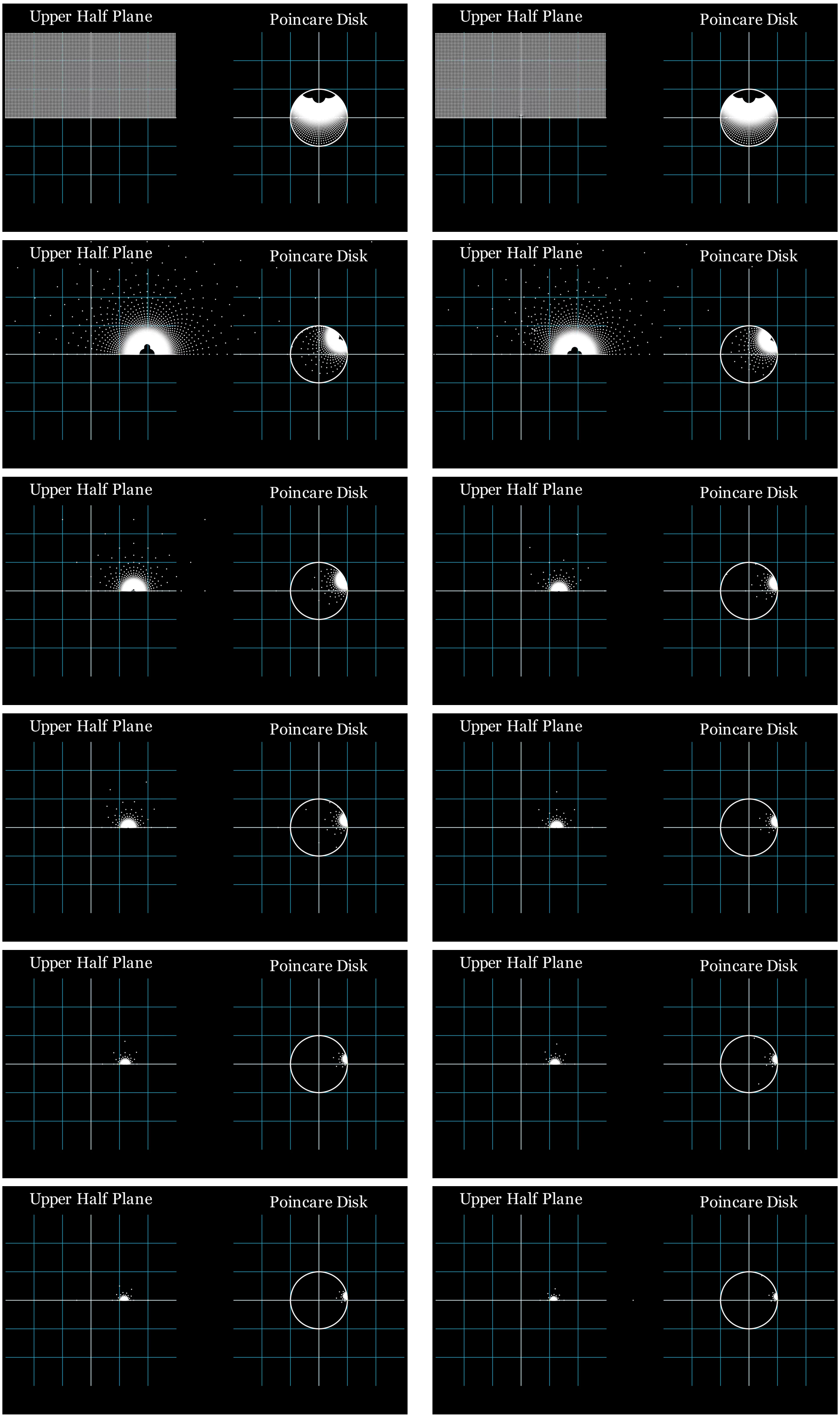
**Theorem 6.1.2** Let $\varphi$ be a parabolic Möbius transformation with a fixed point $p$. Then $H_p$ is invariant under $\varphi$

**Proof (7)**
Let $D = \frac{-1}{z-p}$, $D$ conjugates $B$ to $A$, namely, $A(z) = DBD^{-1}(z)$. By **Prop 6.0.1**, $A$ is also a parabolic transformation, with a fixed point at infinity and thus fixes $H_\infty$. Let $H_\infty$ be a horoball with $\partial H_\infty = \{z \in \mathbb{C} \mid \text{Im}(z) = t_0\}$. $D^{-1}$ maps $\partial H_\infty$ to $C_p$ and $\mathbb{R}$ to $\mathbb{R}$. Since $\partial H_\infty$ and $\mathbb{R}$ are tangent at $\infty$, $C_p$ is tangent to $\mathbb{R}$ at $p$. Therefore $H_p = D^{-1}(H_\infty)$ is a horoball at $p$, and since $A$ leaves $H_\infty$ invariant, we conclude that $B$ leaves $H_p$ invariant.

**Example 6.1 — Parabolic.**
Let $\varphi : \mathbb{H} \mapsto \mathbb{H}$ be defined by

$$\varphi(z) = \frac{2z-1}{z}$$

$\varphi$ is parabolic since $(\text{tr}\,\varphi)^2 = 4$.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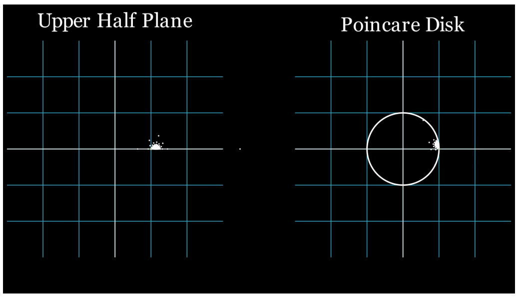
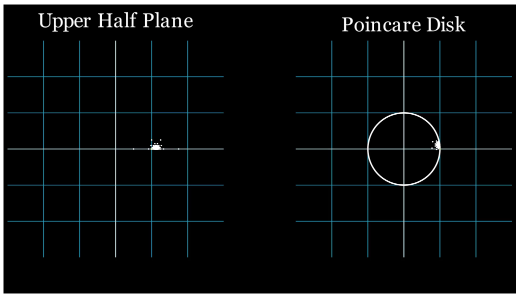



## 6.2 Elliptic

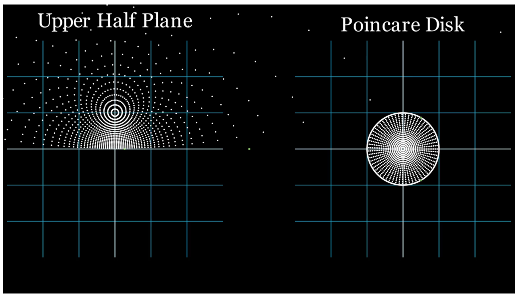
**Definition 6.2.1** A Möbius transformation $\varphi$ is Elliptic if $\varphi$ has two complex fixed points.

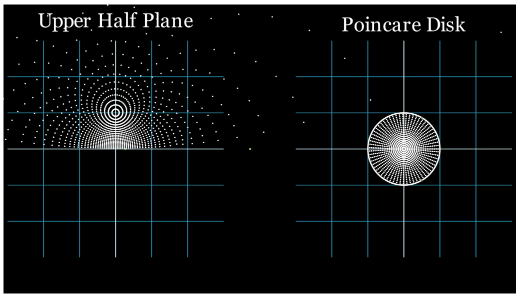
Equivalently, $(a+d)^2 \in [0,4)$. Since the coefficients are real, we cannot have one complex root. Applying the conjugate root theorem, we have one fixed point in $\mathbb{H}$ and the other is outside $\mathbb{H}$. Geometrically elliptic transformations are rotations around the fixed point.

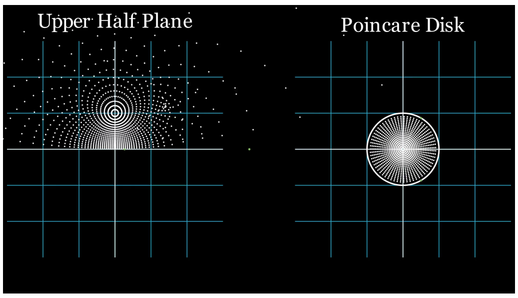
**Example 6.2 — Elliptic.**
Let $\varphi : \mathbb{D} \mapsto \mathbb{D}$ be defined by

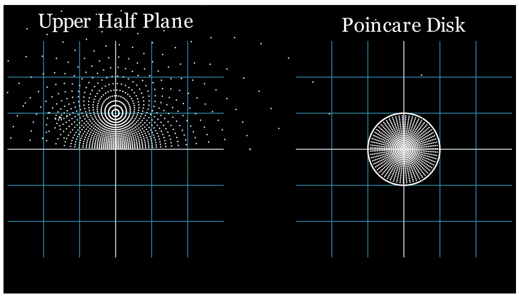
$$\varphi(z) = iz$$

$\varphi$ is elliptic since it has one fixed point in $\mathbb{D}$. Visually, we don't see the points moving since we are rotating by $\frac{\pi}{2}$ radians each iteration.



## 6.3 Hyperbolic

**Definition 6.3.1** A Möbius transformation $\varphi$ is Hyperbolic if $\varphi$ has two distinct real fixed points.

Equivalently, $(a+d)^2 > 4$. An example of a hyperbolic transformation is the dilation $z \to 2z$ in $\mathbb{H}$. Geometrically hyperbolic transformations are dilations in the upper half complex plane.

**Proposition 6.3.1** Let $A(z) = \lambda z$ for $\lambda > 1$ and define,

$$\ell(A) = \inf_{z\in\mathbb{H}} d_{\mathbb{H}}(z, A(z))$$

Then inf is achieved for $z \in \text{Im}$.

**Proof (7)**
Let $z \in \mathbb{H}$. The radial projections of $z$ and $A(z)$ onto the imaginary axis are $|z|i$ and $\lambda|z|i$. Let $\phi_p(w) = \frac{w-p}{w-\bar{p}}$. $\phi_p$ maps the upper half plane to the unit disk sending $p \to 0$.

$$\begin{aligned}
d_{\mathbb{H}}(z,\lambda z) &= d_{\mathbb{D}}\left(0, \frac{(\lambda-1)z}{\lambda z - \bar{z}}\right) \\
&= \ln \frac{1 + \frac{|(\lambda-1)z|}{|\lambda z - \bar{z}|}}{1 - \frac{|(\lambda-1)z|}{|\lambda z - \bar{z}|}} \\
&= \ln \frac{|\lambda z - \bar{z}| + |(\lambda-1)z|}{|\lambda z - \bar{z}| - |(\lambda-1)z|} \\
&\geq \ln \frac{|\lambda z| + |\bar{z}| + |(\lambda-1)z|}{|\lambda z| + |\bar{z}| - |(\lambda-1)z|} \\
&= \ln \lambda \\
&= d_{\mathbb{H}}(i, \lambda i) \\
&= d_{\mathbb{H}}(|z|i, \lambda|z|i)
\end{aligned}$$

Therefore the distance between $z$ and $\lambda z$ has a minimum of the distance between their radial projections onto the imaginary axis. Equality occurs when $|\lambda z - \bar{z}| = |\lambda z| + |\bar{z}|$.

$$\begin{aligned}
|\lambda z - \bar{z}|^2 &= \lambda^2|z|^2 - 2\lambda\,\text{Re}\{z^2\} + |\bar{z}|^2 \\
(|\lambda z| + |\bar{z}|)^2 &= (\lambda+1)^2|z|^2 \\
\text{Re}(z^2) &= |z|^2
\end{aligned}$$

Therefore $\text{Re}(z) = 0$, so the minimum is achieved on the imaginary axis.

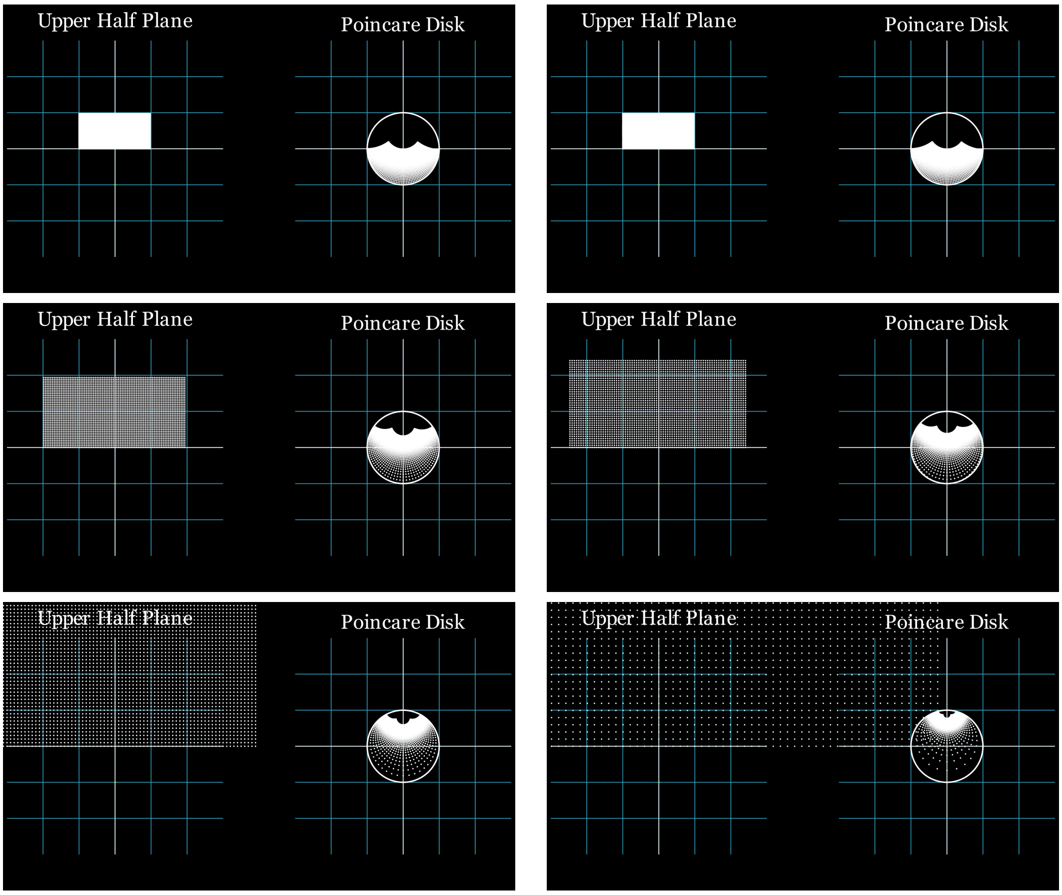
**Theorem 6.3.2** Let $B$ be a hyperbolic Möbius transformation with fixed points $(u,v)$ and $L(t)$ be the geodesic in $\mathbb{H}$ between $u$ and $v$. Then $\ell(B)$ achieves a minimum iff $z \in \ell(B)$

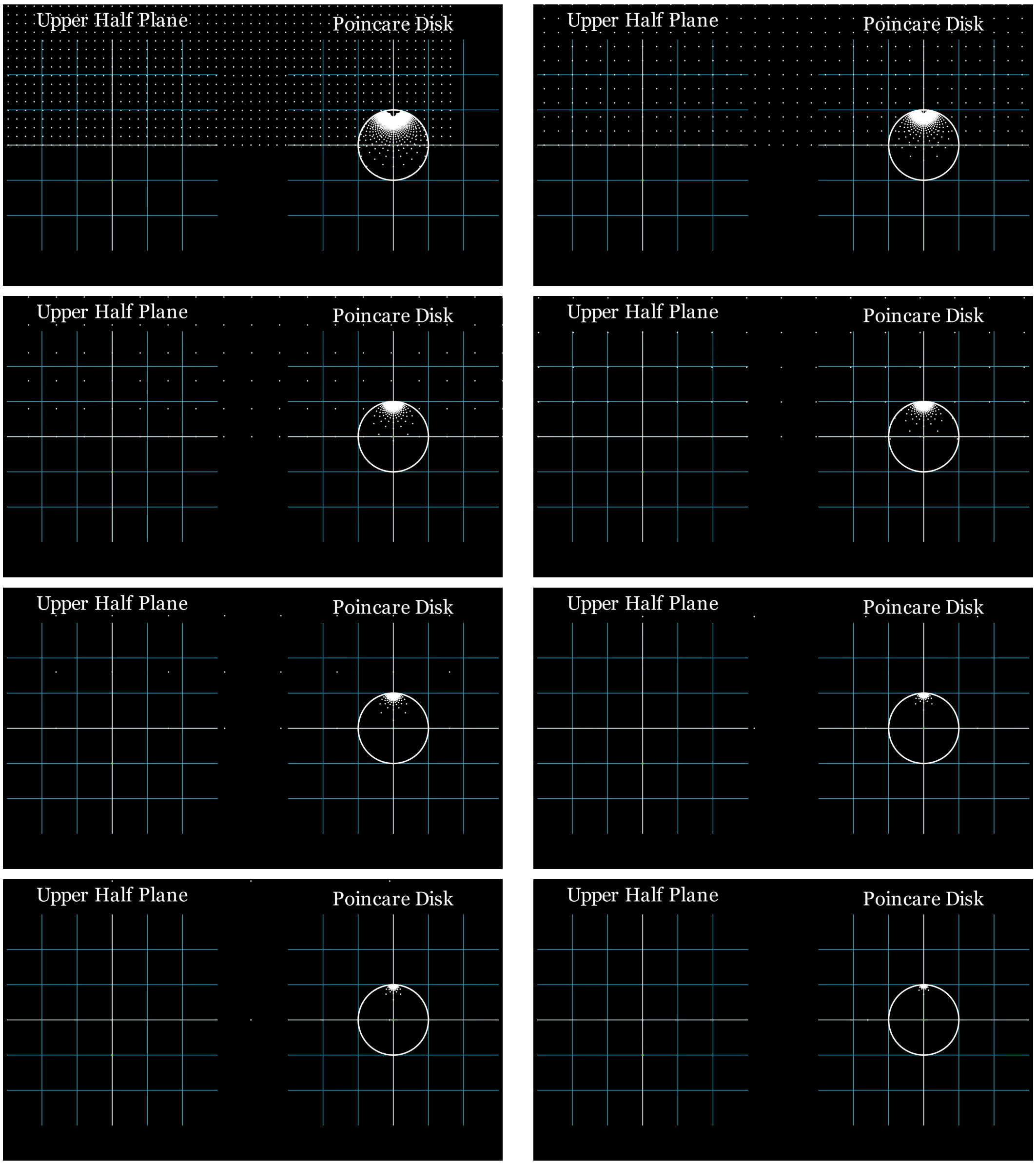
**Proof (7)**
We can assume $u > v$. Let $D(z) = \frac{z-u}{z-v}$, then $D$ takes the fixed points of $B$, $u$, $v$ to 0 and $\infty$ respectively. $D$ is also an isometry of $\mathbb{H}$ and maps $L$ to Im. Thus we have $BDB^{-1}(z) = \lambda z$ for some $\lambda > 0$, so we apply the previous proposition to $BDB^{-1}$ if $\lambda > 1$ and $BD^{-1}B^{-1}$ if $\lambda < 1$. Since the identity is not hyperbolic, $\lambda \neq 1$.

**Example 6.3 — Hyperbolic.**
Let $\varphi : \mathbb{H} \mapsto \mathbb{H}$ be defined by

$$\varphi(z) = 2z$$

$\varphi$ is parabolic since $(\operatorname{tr}\varphi)^2 = 9$.

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# IV Denjoy Wolff Theorem

# 7. Julia's Lemma

In this section we prove Julia's lemma, a more general theorem than the Denjoy Wolff theorem, which we will use to prove the Denjoy Wolff theorem. The main type of set we will be dealing with throughout this chapter is called a horocycle.

**Definition 7.0.1 — Horocycle.**
Let $\tau \in \partial\mathbb{D}$, and $R > 0$. The horocycle $E(\tau, R)$ is,

$$E(\tau,R) = \{z \in \mathbb{D} \mid \frac{|\tau - z|^2}{1-|z|^2} < R\}$$

$E(\tau,R)$ is an open Euclidean disk tangent to the unit disk at $\tau$ of radius $R/(R+1)$. To begin we prove a property of points inside a horocycle.

**Proposition 7.0.1** $z \in E(\tau,R)$ iff

$$\liminf_{w\to\tau} d_{\mathbb{D}}(z,w) - d_{\mathbb{D}}(0,w) < \ln R$$

**Proof**
Let $z \in \mathbb{D}$, firstly we use $T_z$ to map $z \to 0$ and then apply **Example 3.1**.

$$\begin{aligned} d_{\mathbb{D}}(z,w) - d_{\mathbb{D}}(0,w) &= d_{\mathbb{D}}(0,T_z(w)) - d_{\mathbb{D}}(0,w) \\ &= \ln\left(\frac{1+|T_z(w)|}{1-|T_z(w)|}\right) - \ln\left(\frac{1+|w|}{1-|w|}\right) \\ &= \ln\left(\frac{1-|w|}{1-|T_z(w)|}\right) - \ln\left(\frac{1+|w|}{1+|T_z(w)|}\right) \end{aligned}$$

Since $|w| \to 1$ as $w \to \tau$,

$$\lim_{w\to\tau} \ln\left(\frac{1+|w|}{1+|T_z(w)|}\right) = 0$$

This allows us to replace the minus sign above with a positive one,

$$\begin{aligned}\liminf_{w\to\tau} d_{\mathbb{D}}(z,w)-d_{\mathbb{D}}(0,w) &= \liminf_{w\to\tau}\ln\left(\frac{1-|w|}{1-|T_z(w)|}\right)+\ln\left(\frac{1+|w|}{1+|T_z(w)|}\right)\\ &= \liminf_{w\to\tau}\ln\left(\frac{1-|w|^2}{1-|T_z(w)|^2}\right)\end{aligned}$$

Simplifying the argument of the logarithm,

$$\begin{aligned}\frac{1-|w|^2}{1-|T_z(w)|^2} &= (1-|w|^2)\frac{|1-\bar{z}w|^2}{|1-\bar{z}w|^2-|z-w|^2}\\ &= \frac{(1-|w|^2)|1-\bar{z}w|^2}{1-2\operatorname{Re}(\bar{z}w)+|z|^2|w|^2-(|z|^2-2\operatorname{Re}(\bar{z}w)+|w|^2)}\\ &= \frac{(1-|w|^2)|1-\bar{z}w|^2}{(1-|w|^2)(1-|z|^2)}\\ &= \frac{|1-\bar{z}w|^2}{1-|z|^2}\\ &= \frac{|\bar{w}-|w|^2\bar{z}|^2}{|w|^2(1-|z|^2)}\\ &= \frac{|w-|w|^2z|^2}{|w|^2(1-|z|^2)}\end{aligned}$$

Since $|w|\to 1$ as $w\to\tau$,

$$\liminf_{w\to\tau} d_{\mathbb{D}}(z,w)-d_{\mathbb{D}}(0,w) = \ln\left(\frac{|\tau-z|^2}{1-|z|^2}\right)$$

So the proposition follows from the definition of a horocycle.

The next notion we need is called the boundary dilation coefficient which measures how quickly $\varphi$ moves points towards $\sigma$.

**Definition 7.0.2 — Boundary Dilation Coefficient.**
Let $\varphi:\mathbb{D}\mapsto\mathbb{D}$ be holomorphic and $\sigma\in\partial\mathbb{D}$, then

$$\alpha_\varphi(\sigma)=\liminf_{z\to\sigma}\frac{1-|\varphi(z)|}{1-|z|}$$

is the boundary dilation coefficient.

Similarly to the horocycle, we prove a property of the boundary dilation coefficient.

**Proposition 7.0.2**

$$\liminf_{w\to\sigma} d_{\mathbb{D}}(0,w)-d_{\mathbb{D}}(0,\varphi(w)) = \ln\alpha_\varphi(\sigma)$$

**Proof**
Using **Example 3.1**,

$$d_{\mathbb{D}}(0,w) - d_{\mathbb{D}}(0,\varphi(w)) = \ln\left(\frac{1+|w|}{1-|w|}\right) - \ln\left(\frac{1+|\varphi(w)|}{1-|\varphi(w)|}\right)$$
$$= \ln\left(\frac{1-|\varphi(w)|}{1-|w|}\right) + \ln\left(\frac{1+|w|}{1+|\varphi(w)|}\right)$$

Therefore,

$$\liminf_{w\to\sigma} d_{\mathbb{D}}(0,w) - d_{\mathbb{D}}(0,\varphi(w)) = \ln \alpha_{\varphi}(\sigma)$$

The final proposition we need before we can prove Julia's Lemma involves a property of the hyperbolic metric.

**Proposition 7.0.3** Let $\varphi : \mathbb{D} \mapsto \mathbb{D}$ be holomorphic, then $\varphi$ contracts distances with respect to the hyperbolic metric $d_{\mathbb{D}}$. Namely,

$$d_{\mathbb{D}}(\varphi(u),\varphi(v)) \leq d_{\mathbb{D}}(u,v)$$

for $u,v \in \mathbb{D}$. Equality occurs iff $\varphi$ is an automorphism of the unit disk

**Proof**
Equality follows from the construction of $d_{\mathbb{D}}$,

$$d_{\mathbb{D}}(\varphi(u),\varphi(v)) = d_{\mathbb{D}}(0,T_{\varphi(u)}(\varphi(v)))$$
$$= \ln\left(\frac{1+|T_{\varphi(u)}(\varphi(v))|}{1-|T_{\varphi(u)}(\varphi(v))|}\right)$$

By the Schwarz Pick Lemma

$$\leq \ln\left(\frac{1+|T_u(v)|}{1-|T_u(v)}\right)$$
$$= d_{\mathbb{D}}(u,v)$$

**Theorem 7.0.4 — Julia's Lemma.**
Let $\varphi : \mathbb{D} \mapsto \mathbb{D}$ be holomorphic and $\sigma\partial\mathbb{D}$. Assume $\alpha_{\varphi}(\sigma) < \infty$, then there exists $\eta \in \mathbb{D}$ such that for all $R > 0$,

$$\varphi(E(\sigma,R)) \subset E(\eta,\alpha_{\varphi}(\sigma)R)$$

**Julia's Lemma Proof**

Since $\alpha_\varphi(\sigma) < \infty$ we can find a sequence $(w_1, w_2, \ldots)$ converging to $\sigma$ such that,

$$\lim_{k\to\infty} \frac{1-|\varphi(w_k)|}{1-|w_k|} = \alpha_\varphi(\sigma)$$

Using **Prop 7.0.2**,

$$\lim_{k\to\infty} d_{\mathbb{D}}(0,w_k) - d_{\mathbb{D}}(0,\varphi(w_k)) = \ln \alpha_\varphi(\sigma)$$

Let $\eta = \lim_{k\to\infty} \varphi(w_k)$ and $z \in E(\sigma, R)$. We want to show that $\varphi(z) \in E(\eta, \alpha_\varphi(\sigma)R)$. By **Prop 7.0.1** this is equivalent to

$$\lim_{k\to\infty} d_{\mathbb{D}}(\varphi(z),\varphi(w_k)) - d_{\mathbb{D}}(0,\varphi(w_k)) < \ln\big(\alpha_\varphi(\sigma)R\big)$$

By **Prop 7.0.3**,

$$\begin{aligned} d_{\mathbb{D}}(\varphi(z),\varphi(w_k)) - d_{\mathbb{D}}(0,\varphi(w_k)) &< d_{\mathbb{D}}(z,w_k) - d_{\mathbb{D}}(0,\varphi(w_k)) \\ &= d_{\mathbb{D}}(z,w_k) - d_{\mathbb{D}}(0,w_k) + d_{\mathbb{D}}(0,w_k) - d_{\mathbb{D}}(0,\varphi(w_k)) \end{aligned}$$

By **Prop 7.0.1**, $z \in E(\sigma,R)$ is equivalent to

$$\lim_{k\to\infty} d_{\mathbb{D}}(z,w_k) - d_{\mathbb{D}}(0,w_k) < \ln R$$

By **Prop 7.0.2**, $d_{\mathbb{D}}(0,w_k) - d_{\mathbb{D}}(0,\varphi(w_k)) \to \ln \alpha_\varphi(\sigma)$ as $k \to \infty$. Therefore,

$$\lim_{k\to\infty} d_{\mathbb{D}}(\varphi(z),\varphi(w_k)) - d_{\mathbb{D}}(0,\varphi(w_k)) < \ln R + \ln \alpha_\varphi(\sigma)$$

and,

$$\varphi(E(\sigma,R)) \subset E(\eta, \alpha_\varphi(\sigma)R)$$

# 8. Denjoy-Wolff Theorem

The following proposition and Denjoy-Wolff Theorem will allow us to give an generalisation of the classification developed in the previous chapter on Möbius Transformations.

**Proposition 8.0.1** Let $\phi : \mathbb{D} \mapsto \mathbb{D}$ be holomorphic, not an automorphism. Suppose there exists $\tau \in \mathbb{D}$ such that $\phi(\tau) = \tau$. Then $\phi^{\circ n}$ converges uniformly on compacta to the constant map $z \mapsto \tau$.

**Proof**

Firstly suppose $\phi(0) = 0$. Then by the Schwarz lemma we have that $|\phi(z)| \leq |z|$. Let $0 < r < 1$ and $M(r) = \max_{|z| \leq r} |\phi(z)|$. Let $\delta = M(r)/r > 0$. The previous note ensures $\delta < 1$. Let $\psi = \frac{\phi(rz)}{M(r)}$. Clearly $\psi$ fixes 0 and is holomorphic on $\mathbb{D}$. Since $r \in (0,1)$, $\psi$ is continuous on $\overline{\mathbb{D}}$. Thus by the Schwarz lemma $|\psi(z)| \leq |z|$, which implies that,

$$|\phi(z)| = M(r)\left|\psi\left(\frac{z}{r}\right)\right| \leq \frac{M(r)}{r}|z| \leq \delta|z|$$

Thus by induction we have $|\phi^{\circ n}(z)| \leq \delta^n |z| \leq \delta^n$. Since $\delta < 1$, $\phi^{\circ n}$ tends to 0 uniformly on $r\overline{\mathbb{D}}$ and since $r$ was arbitrary, $\phi^{\circ n}$ converges uniformly to 0 on compact subsets of $\mathbb{D}$.

If $\tau \neq 0$ then we apply the previous argument to $\varphi = T_\tau \circ \phi \circ T_\tau$. Then $\varphi$ is holomorphic, not an automorphism and thus $\varphi^{\circ n}$ converges to 0 uniformly on compacta. Hence $\phi^{\circ n} = T_\tau \circ \varphi^{\circ n} \circ t_\tau$ converges uniformly on compacta to $T_\tau(0) = \tau$.

Before we state the Denjoy Wolff Theorem, we make some remarks about Möbius Transformations.

Let $T$ be a parabolic transformation in $\mathbb{H}$. We found that one could conjugate this transform using another automorphism $D$ which takes the fixed point of $T$ to $\infty$. Then $D \circ T \circ D^{-1}$ is a parabolic transformation with a fixed point at $\infty$ and thus a translation. So $(D \circ T \circ D^{-1})^{\circ n}(z) \to \infty$ as $n \to \infty$ and thus $T^{\circ n}$ converges uniformly on compacta to $z \mapsto p$ where $p$ is the fixed point of $T$.

Similarly let $T$ be a hyperbolic transformation, and $D$ be an automorphism conjugating the fixed points of $T$ to 0 and $\infty$. Then $D \circ T \circ D^{-1}$ is a hyperbolic transformation with fixed points at 0 and $\infty$ and thus a dilation. So $(D \circ T \circ D^{-1})^{\circ n}(z) \to \infty$ (or 0 depending on the magnitude of the dilation) as $n \to \infty$ and thus $T^{\circ n}$ converges uniformly on compacta to $z \mapsto p$ where $p$ is a fixed point of $T$ (the fixed point corresponding to 0 or $\infty$, which depends on the previous statement). We are now ready to state the Denjoy Wolff Theorem.

**Theorem 8.0.2 — Denjoy-Wolff Theorem.**
Let $\varphi : \mathbb{D} \mapsto \mathbb{D}$ be holomorphic and assume $\varphi$ has no fixed points in $\mathbb{D}$. Then there exists $\tau \in \partial\mathbb{D}$ unique such that $\alpha_\varphi(\tau) \leq 1$ and for every $R > 0$,

$$\varphi(E(\tau,R)) \subset E(\tau,R) \tag{8.1}$$

**Denjoy-Wolff Theorem Proof**
Firstly for each $z \in \mathbb{D}$, the sequence $|\varphi^{\circ n}(z)|$ converges to 1. Suppose not, then there exists a subsequence $n_k$ such that $\lim_{k\to\infty} \varphi^{\circ n_k}(z) = p \in \mathbb{D}$. Since elliptic transformation have fixed points in $\mathbb{D}$ and from the previous discussion on uniform convergence, $\varphi$ is not an automorphism. By **Proposition 7.0.3**, the map $\mathbb{N} \ni n \mapsto d(\varphi^{\circ n}(z_0), \varphi^{\circ(n+1)}(z_0))$ is decreasing and thus converges. Since $p = \lim_{k\to\infty} \varphi^{\circ n_k}(z)$,

$$\begin{aligned} d(p,\varphi(p)) &= \lim_{k\to\infty} d(\varphi^{\circ n_k}(z_0), \varphi^{\circ(n_k+1)}(z_0)) \\ &= \lim_{k\to\infty} d(\varphi^{\circ n_k+1}(z_0), \varphi^{\circ(n_k+2)}(z_0)) \\ &= d(\varphi(p), \varphi^{\circ 2}(p)) \end{aligned}$$

Since $\varphi$ is not an automorphism, this is only possible if $\varphi(p) = p$, a contradiction. Therefore, for every $z \in \mathbb{D}$, $\varphi^{\circ n}(z)$ accumulates on the boundary of $\mathbb{D}$. Let $(w_n)_{n\in\mathbb{N}}$ be a sequence defined by $w_{n+1} = \varphi(w_n)$ and $w_1 = 0$. Since $\lim_{n\to\infty} |w_n| = 1$, there exists a subsequence $n_k$ such that $|\varphi(w_{n_k})| > |w_{n_k}|$. Thus up to extracting a converging subsequence, we can assume $w_{n_k}$ converges to $\tau \in \mathbb{D}$. By construction, $\varphi(w_{n_k})$ also converges to $\tau$. Since $|\varphi(w_{n_k})| \geq |w_{n_k}|$, then $1 - |\varphi(w_{n_k})| < 1 - |w_{n_k}|$ and thus,

$$\alpha_\varphi(\sigma) = \lim_{n\to\infty} \frac{1 - |\varphi(w_n)}{1 - |w_n|} \leq 1$$

Therefore by Julia's Lemma, for all $R > 0$

$$\varphi(E(\tau,R)) \subset E(\tau,R)$$

Suppose there exists $\tau' \in \mathbb{D}$ that also satisfies the [8.1]. Let $R, R' \in \mathbb{R}$ such that $\overline{E(\tau,R)} \cap \overline{E(\tau',R')} = z_0$, $R$ and $R'$ exists and are well defined as $E$ is an open disk in $\mathbb{R}^2$. Thus,

$$\begin{aligned} \varphi(z_0) &= \varphi(\overline{E(\tau,R)} \cap \overline{E(\tau',R')}) \\ &= \varphi(\overline{E(\tau,R)}) \cap \varphi(\overline{E(\tau',R')}) \end{aligned}$$

Since $\varphi$ is continuous,

$$\begin{aligned} &\subset \overline{\varphi(E(\tau,R))} \cap \overline{\varphi(E(\tau',R'))} \\ &\subset \overline{E(\tau,R)} \cap \overline{E(\tau',R')} \\ &= z_0 \end{aligned}$$

Which is a contradiction, $z_0 \in \mathbb{D}$ and $\varphi$ has no fixed points in $\mathbb{D}$. Therefore $\tau$ is unique.

Finally we show that $\varphi$ converges uniformly on compacta to $z \mapsto \tau$. By Vitali's Theorem [9], it suffices to show $\lim_{n\to\infty} \varphi^{\circ n}(z_0) = \tau$. Let $z_0 \in \mathbb{D}$. Then there exists $R > 0$ such that $z_0 \in E(\tau,R)$. By [8.1], $\varphi^{\circ n}(z_0) \in E(\tau,R)$. Since $\varphi^{\circ n}(z_0)$ accumulates on $\partial\mathbb{D}$ and $\overline{E(\tau,R)} \cap \partial\mathbb{D} = \tau$, $\lim_{n\to\infty} \varphi^{\circ n}(z_0) = \tau$.

**Definition 8.0.1** Let $\varphi : \mathbb{D} \mapsto \mathbb{D}$ be holomorphic, not the identity.

1. If $\varphi$ has a fixed point in $\mathbb{D}$, then its unique fixed point is called the *Denjoy-Wolff point* of $\varphi$.
2. If $\varphi$ has no fixed points in $\mathbb{D}$, then the unique point $\tau \in \partial\mathbb{D}$ given by the Denjoy-Wolff Theorem is the *Denjoy-Wolff point* of $\varphi$

Moreover, $\varphi$ is

1. *elliptic* if its Denjoy-Wolff point is in $\mathbb{D}$,
2. *hyperbolic* if its Denjoy-Wolff point $\tau$ belongs to $\partial\mathbb{D}$ and $\alpha_\varphi(\tau) \in (0,1)$,
3. *parabolic* if its Denjoy-Wolff point $\tau$ belongs to $\partial\mathbb{D}$ and $\alpha_\varphi(\tau) = 1$.

This definition generalizes the classification made in the previous chapter from automorphisms to holomorphic self maps.

# V Semigroups

# 9. Preliminaries

Before we can study semigroups, we need some definitions from topology and algebra

## 9.1 Topology

**Definition 9.1.1 — Topology.**
A *topology* of a set $X$ is a collection $U$ of subsets of $X$ called *open sets* such that,

1. An arbitrary union of elements of $U$ is an element of $U$
2. A finite intersection of elements of $U$ is an element of $U$
3. $X$ and $\emptyset$ are elements of $U$

A special type of subset of a topological space is a compact set,

**Definition 9.1.2 — Compact Set.**
Let $X$ be a topological space. A set $K \subset X$ is *compact* if any open cover has a finite sub cover. More explicitly for any collection $C = \{U_1, U_2, \ldots\}$ of open sets in $X$ such that

$$K \subset \bigcup_{U \in C} U$$

there exists a finite subset of $C$, $\{U_1, \ldots, U_k\}$ such that

$$K \subset \bigcup_{i=1}^{k} U_i$$

Two properties of topological spaces are as follows,

**Definition 9.1.3 — Hausdorff.**
Let $X$ be a topological space. $X$ is *Hausdorff* if for any two points $x, y \in X$ such that $x \neq y$, there exists open sets $U_x, U_y$ containing $x, y$ respectively such that $U_x \cap U_y = \emptyset$.

The definition for a topology above is stronger than required, so we reduced collection of open sets known as a basis,

**Definition 9.1.4 — Base Topology.**
A *base* for a topology on $X$ is a collection of open sets $U'$ such that,

1. A finite intersection of elements of $U'$ is an element of $U'$
2. $X$ and $\emptyset$ are elements of $U'$

We can construct a topology from a base by taking arbitrary unions of elements of $U'$. The final property of topological spaces we are interested in is the size of a basis for a topological space,

**Definition 9.1.5 — Second countable.**
Let $X$ be a topological space. $X$ is *second countable* if $X$ has a countable base.

We can reduce the size of our collection of open sets further to a subbase,

**Definition 9.1.6 — Subbase Topology.**
A *subbase* for a topology on $X$ is a collection of open sets $U''$ such that,

1. $X$ and $\emptyset$ are elements of $U''$

We can construct a base from a subbase by taking finite intersections of elements of $U''$. A subbase allows us to abstract away the axioms for a topology, greatly simplifying the construction and verification of topologies.

**Definition 9.1.7 — Compact-Open Topology.**
Let $X,Y$ be topological spaces and $C(X,Y)$ be the collection of all continuous maps from $X$ to $Y$. Given a compact set $K \subset X$ and an open set $U \subset Y$, let $V(K,U)$ denote the set $\{f \in C(X,Y) \mid f(K) \subset U\}$. The collection $V(K,U)$ is a subbase for the *compact-open topology* on $C(X,Y)$

This is clearly a subbase as $V(K,\emptyset) = \emptyset$ and $V(\emptyset,Y) = C(X,Y)$.

## 9.2 Algebra

**Definition 9.2.1 — Semigroup.**
A *semigroup* is a set $X$ with an associative binary operation $* : X \times X \mapsto X$. Namely, for any $x,y,z \in X$,

$$(x*y)*z = x*(y*z)$$

While semigroups will lead to interesting results in the following sections, we will also want an object with more structure that we can completely characterise.

**Definition 9.2.2 — Group.**
A *group* is a set $X$ with a binary operation $* : X \times X \mapsto X$ such that,

1. (Identity) $e \in X$ such that $x*e = e*x = x$ for all $x \in X$.
2. (Inverse) For all $x \in X$ there exists $y \in X$ such that $x*y = y*x = e$.

3. (Associativity) For all $x, y, z \in X$, $x * (y * z) = (x * y) * z$.

The most general way of mapping between two (semi) groups is as follows,

**Definition 9.2.3 — Homomorphism.**
Let $(G_0, *_0)$ and $(G_1, *_1)$ be a (semi)group. A map $f : G_0 \mapsto G_1$ is a *homomorphism* if for any $x, y \in G_0$,

$$f(x *_0 y) = f(x) *_1 f(y)$$

A stronger map is an isomorphism, which basically says the two objects have the same group structure.

**Definition 9.2.4 — Isomorphism.**
Let $f$ be a homomorphism. Then $f$ is an *isomorphism* if $f$ is bijective (injective and surjective).

A special type of isomorphism is one from an object to itself,

**Definition 9.2.5 — Automorphism.**
Let $f$ be an isomorphism. Then $f$ is an *automorphism* if $G_0 = G_1$ from the definition of a homeomorphism.

# 10. Semigroups in the Unit Disk

In the previous chapter we iterated Möbius transformations by repeatedly composing them. This is essentially a map $\mathbb{N} \ni n \mapsto \varphi^{\circ n}$. Now we extended this idea to real numbers and general holomorphic self-maps of the unit disc. This chapter follows Chapter 8 of [4].

**Definition 10.0.1 — Semigroup in the Unit Disk.**
An *algebraic semigroup* $(\varphi_t)$ of holomorphic self maps of the unit disk is a homomorphism between the additive semigroup of non negative real numbers and the composition semigroup of all holomorphic self maps of the unit disk. Namely,

1. $\varphi_t \in \mathrm{Hol}(\mathbb{D},\mathbb{D})$ for all $t \geq 0$
2. $\varphi_0 = \mathrm{id}_{\mathbb{D}}$
3. $\varphi_{s+t} = \varphi_s \circ \varphi_t$ for all $s,t \geq 0$

**Definition 10.0.2 — Continuity.**
An algebraic semigroup $(\varphi_t)$ is continuous if the map

$$[0,\infty) \ni t \mapsto \varphi_t \in \mathrm{Hol}(\mathbb{D},\mathbb{D})$$

where $[0,\infty)$ has the usual topology and $Hol(\mathbb{D},\mathbb{D})$ has the compact-open topology.

Both of these definitions translate directly to Riemann surfaces.

**Proposition 10.0.1** Let $h$ be a biholomorphism from $\mathbb{D}$ onto a domain $\Omega$ of the complex plane. If $(\phi_t), t \geq 0$ is a family of holomorphic self maps of $\Omega$ such that,

1. $\varphi_0 = id_\Omega$
2. $\varphi_{s+t} = \varphi_s \circ \varphi_t$
3. $[0,\infty) \ni t \mapsto \varphi_t \in \mathrm{Hol}(\Omega,\Omega)$ is continuous where $[0,\infty)$ has the unsual topology and $\mathrm{Hol}(\Omega,\Omega)$ has the compact-open topology

Then $\varphi_t^h = h^{-1} \circ \varphi_t \circ h$ is a continuous semigroup in $\mathbb{D}$.

**Proof**
The proof is a direct verification of the semigroup conditions,

1. $\varphi_t$ is the composition holomorphic functions

2. $t = 0$ is the identity

$$\begin{aligned}\varphi_0^h &= h^{-1} \circ \varphi_0 \circ h \\ &= h^{-1} \circ id_\Omega \circ h \\ &= h^{-1} \circ h \\ &= id_\mathbb{D}\end{aligned}$$

3. Homomorphism condition

$$\begin{aligned}\varphi_{t+s}^h &= h^{-1} \circ \varphi_{t+s} \circ h \\ &= h^{-1} \circ \varphi_t \circ \varphi_s \circ h \\ &= h^{-1} \circ \varphi_t \circ h \circ h^{-1} \circ \varphi_s \circ h \\ &= \varphi_t^h \circ \varphi_s^h\end{aligned}$$

Holomorphic maps are continuous and the composition of continuous maps is continuous, thus $\varphi_t^h$ is continuous.

# 11. Groups in the Unit Disk

**Definition 11.0.1 — Group in the Unit Disk.**
An *algebraic group* $(\varphi_t)$ of holomorphic self maps of the unit disk is an algebraic semigroup in $\mathbb{D}$ such that $\varphi_t \in \text{Aut}(\mathbb{D},\mathbb{D})$ for all $t \geq 0$.

Then we define $\varphi_{-t} = (\varphi_t)^{-1}$

The condition of all iterates being automorphisms is a stronger condition than necessary. We can give an equivalent condition as follows.

**Theorem 11.0.1** Let $(\varphi_t)$ be a semigroup in $\mathbb{D}$. $(\varphi_t)$ is a group in $\mathbb{D}$ iff there exists $t > 0$ such that $\varphi_t \in \text{Aut}(\mathbb{D},\mathbb{D})$.

**Proof**
The forward direction follows from the definition of an algebraic group.

$(\Leftarrow)$ Let $(\varphi_t)$ be a semigroup in $\mathbb{D}$ such that $\varphi_{t_0} \in \text{Aut}(\mathbb{D},\mathbb{D})$. Let $0 < s < t_0$, then

1. Surjective
$$\mathbb{D} = \varphi_{t_0}(\mathbb{D}) = \varphi_s(\varphi_{t_0-s}(\mathbb{D})) \subset \varphi_s(\mathbb{D})$$
2. Injective. Let $u, v \in \mathbb{D}$ be distinct. Since $\varphi_{t_0} \in \text{Aut}(\mathbb{D},\mathbb{D})$, $\varphi_{t_0}(u) \neq \varphi_{t_0}(v)$. Suppose $\varphi_s(u) = \varphi_s(v)$,
$$\begin{aligned}\varphi_{t_0}(u) &= \varphi_{t_0-s}(\varphi_s(u)) \\ &= \varphi_{t_0-s}(\varphi_s(v)) \\ &= \varphi_{t_0}(v)\end{aligned}$$
Which contradicts $\varphi_{t_0}$ is injective. Thus $\varphi_s$ is injective

If $s > t_0$ then $s = nt_0 + r$ for some integer $n$ and $0 < r < t_0$, thus the same results apply to $\varphi_s$ and therefore $(\varphi_t)$ is an algebraic group.

Since an element of an algebraic group in an automorphism, we can classify each iterate by their fixed points. We won't prove it here, but a fixed point of one iterate is a common fixed point of all iterates, thus we have the following classification.

**Theorem 11.0.2 — Classification of Groups (7).**
Let $(\varphi_t)$ be a non trivial group in $\mathbb{D}$. Then $(\varphi_t)$ has one of the three following forms,

1. **(Elliptic)** There exists $\tau \in \mathbb{D}$ (Denjoy-Wolff point) and $\omega \in \mathbb{R} - \{0\}$ such that,

$$\varphi_t(z) = \frac{(e^{-i\omega t} - |\tau|^2)z + \tau(1 - e^{-i\omega t})}{\bar{\tau}(e^{-i\omega t} - 1)z + 1 - |\tau|^2 e^{-i\omega t}}$$

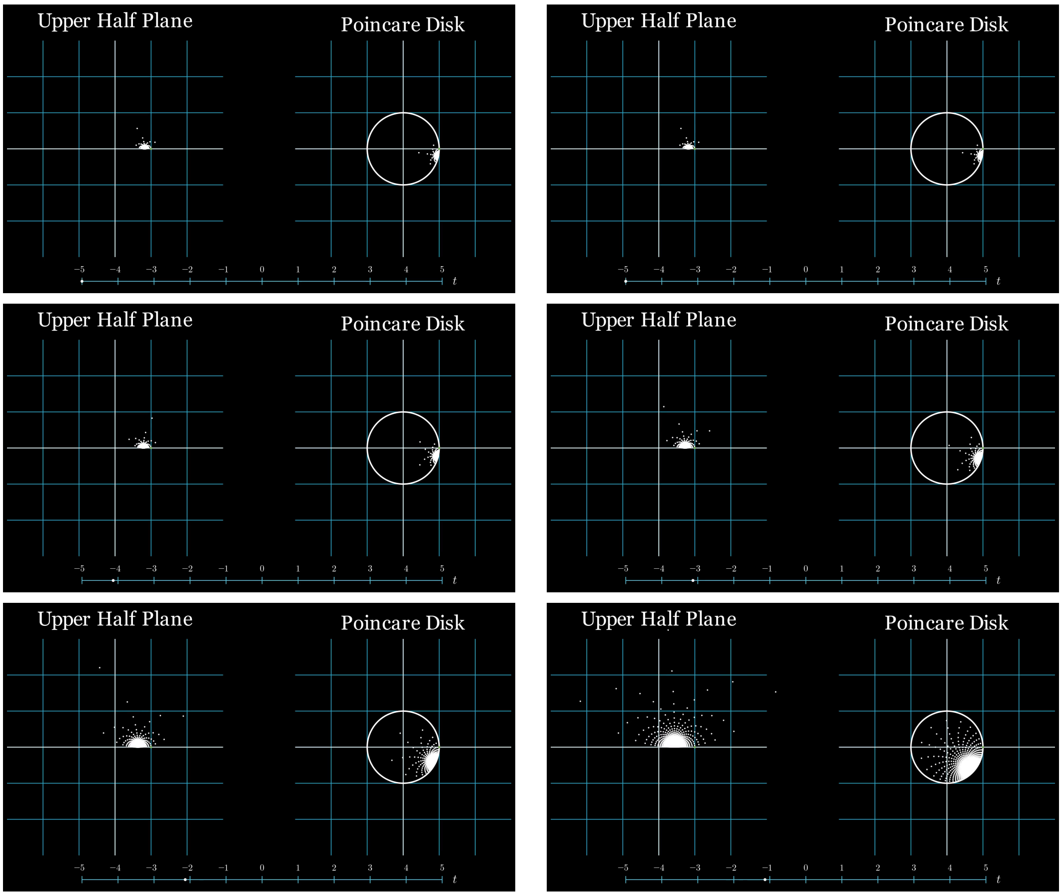
2. **(Parabolic)** There exists $\tau \in \partial\mathbb{D}$ (Denjoy-Wolff point) and $\alpha \in \mathbb{R} - \{0\}$ such that,

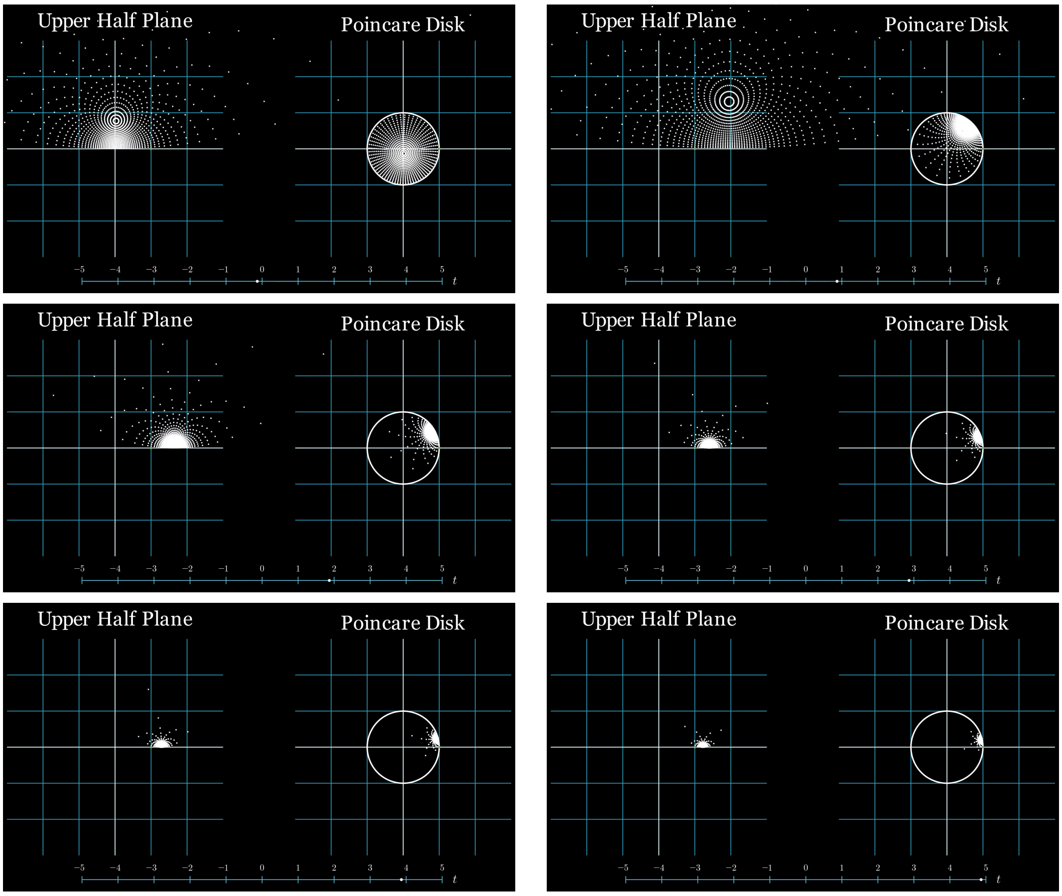
$$\varphi_t(z) = \frac{(1 - i\alpha t)z + i\alpha t\tau}{-i\alpha\bar{\tau}tz + 1 + i\alpha t}$$

3. **(Hyperbolic)** There exists $\tau, \sigma \in \partial\mathbb{D}$ with $\tau$ being the Denjoy-Wolff point, $\sigma$ the other fixed point and $\alpha \in \mathbb{R} - \{0\}$ such that,

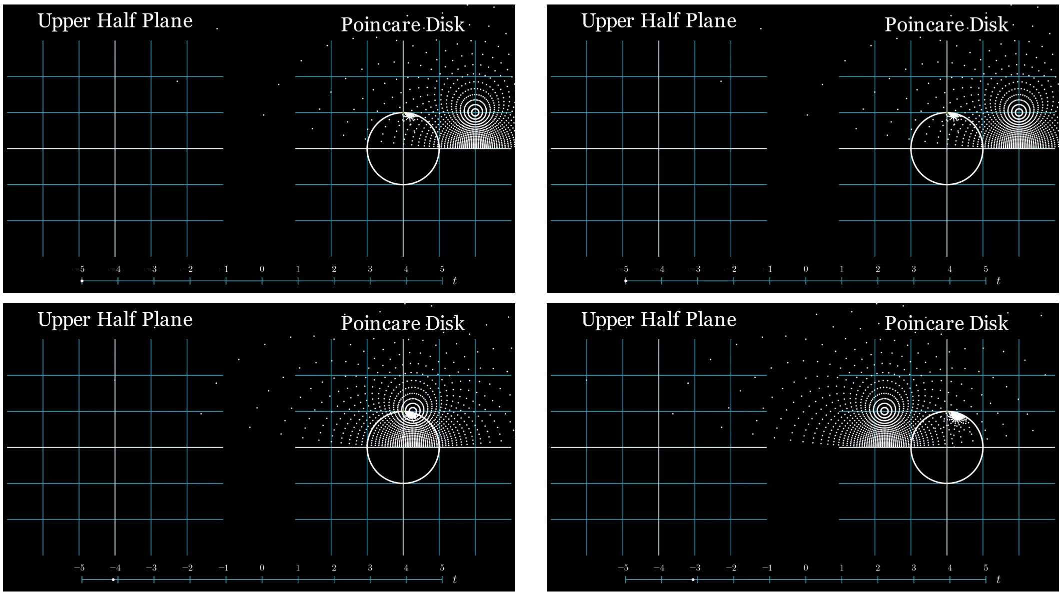
$$\varphi_t(z) = \frac{(\sigma - \tau e^{\alpha t})z + \tau\sigma(e^{\alpha t} - 1)}{(1 - e^{\alpha t})z + \sigma e^{\alpha t} - \tau}$$

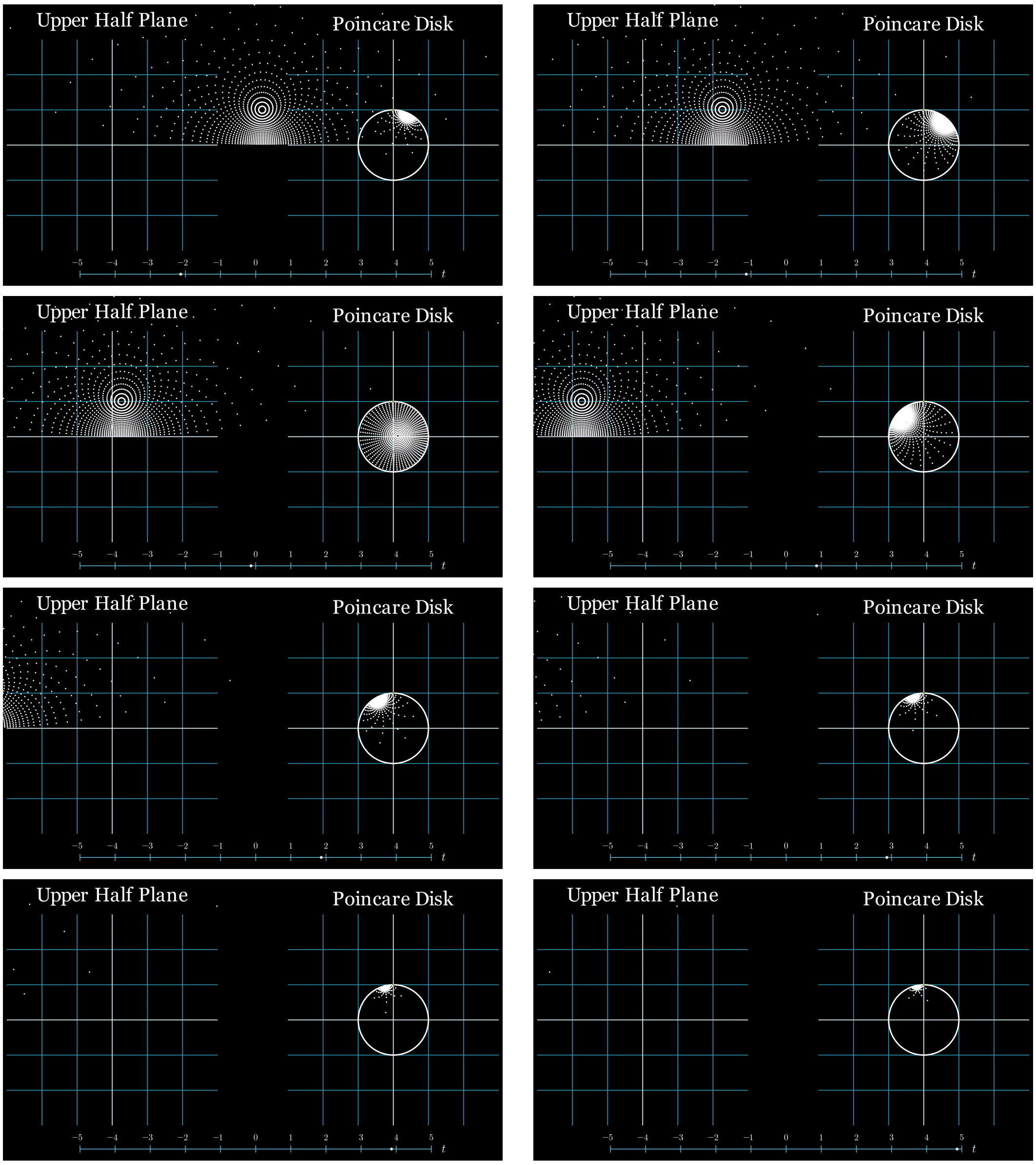
**Example 11.1 — Parabolic.**
Let $\tau = \frac{1}{2}$ and $\alpha = 1$. Iterating $\varphi_t$, on the upper half plane the points are translated along horoballs at the fixed point, moving away from one side of the fixed point towards the other.



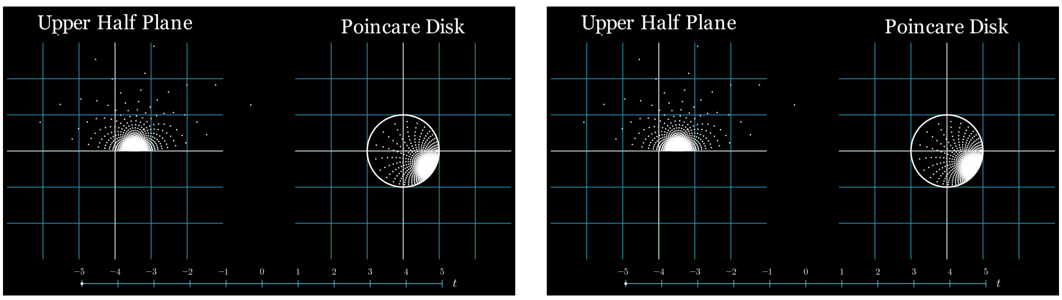


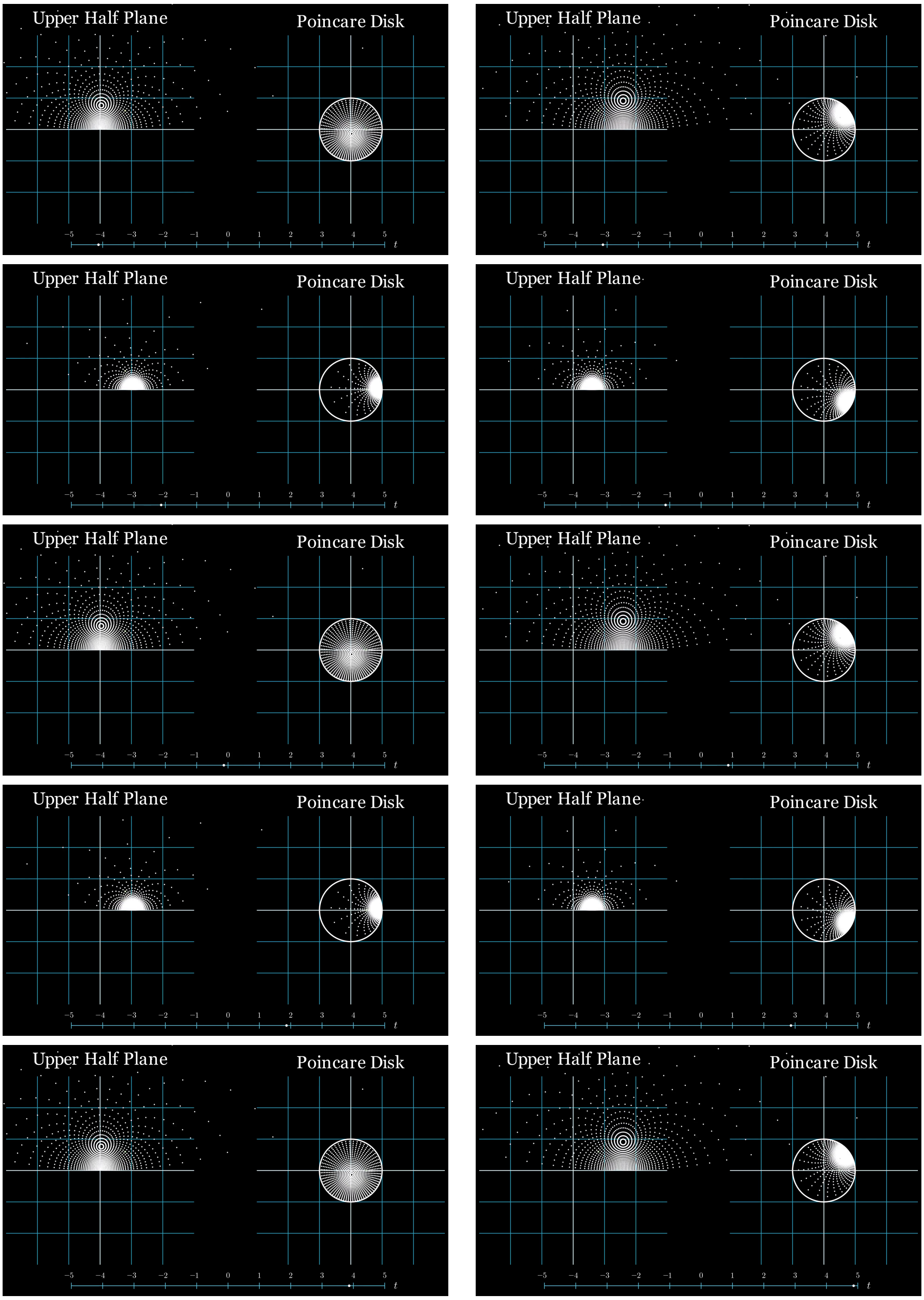

Let $\tau = i$ and $\alpha = 1$. Iterating $\varphi_t$, on the upper half plane the points are translated along horoballs at the fixed point. Since the fixed point is at $\infty$, the horoballs are horizontal lines, so the points are simply translated in the plane.



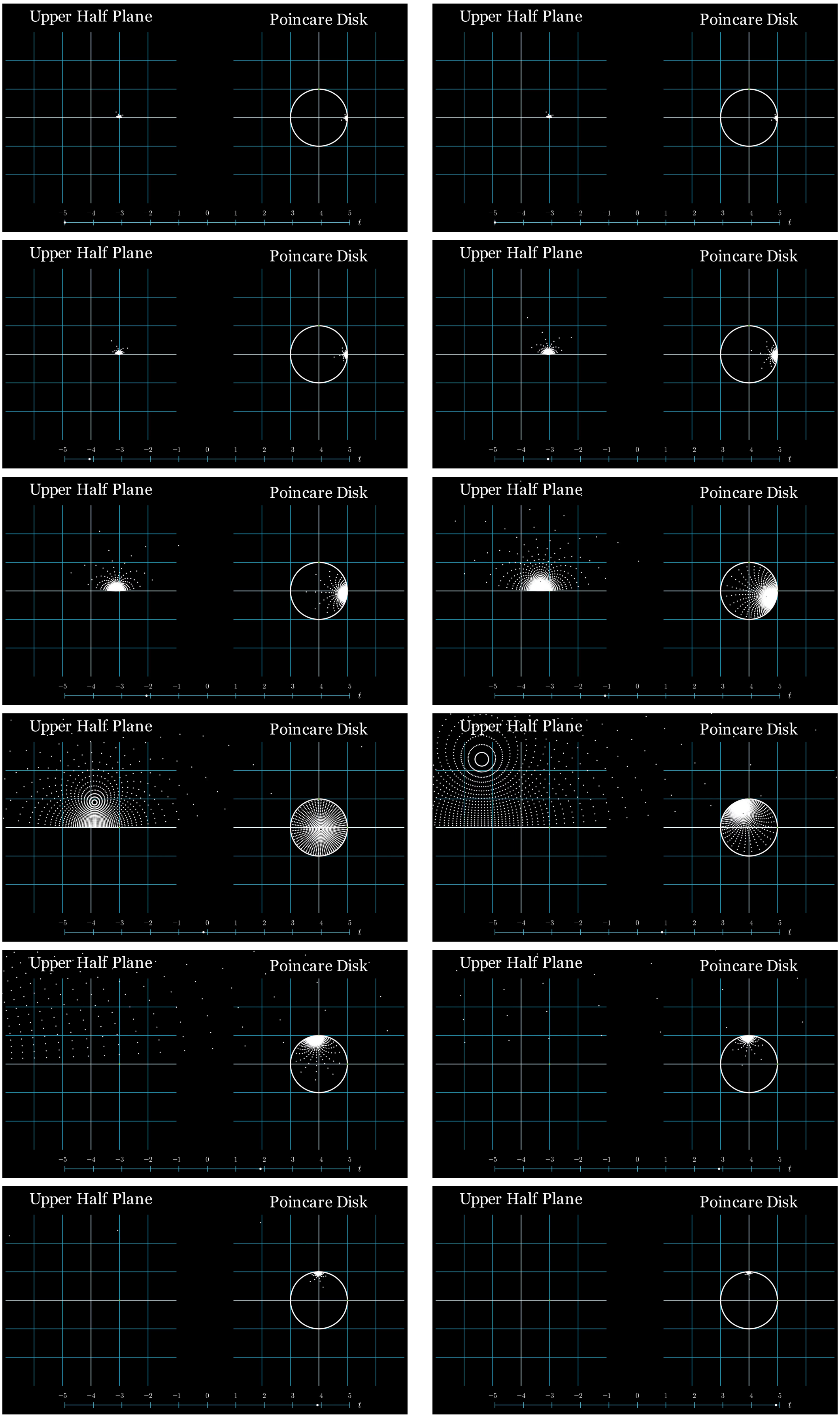



**Example 11.2 — Elliptic.**
Let $\tau = 0.5$ and $\omega = \frac{\pi}{2}$. Iterating $\varphi_t$, the points orbit around the fixed point, 0.5.

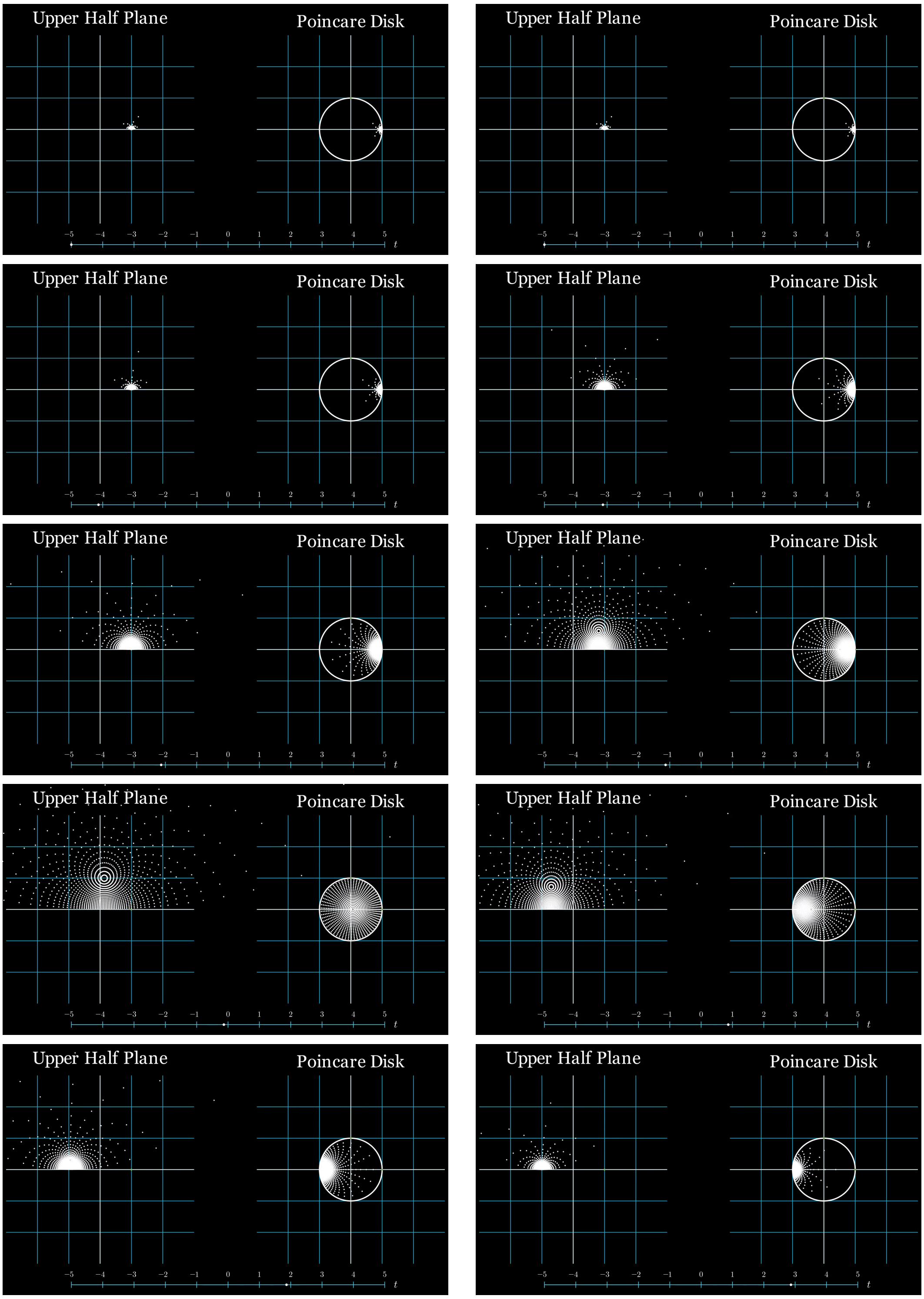


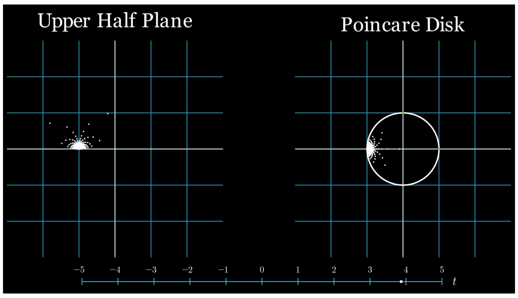


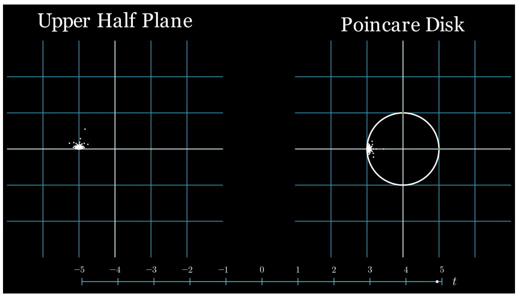

**Example 11.3 — Hyperbolic.**
Let $\sigma = 0$, $\tau = i$ and $\alpha = 1$. Iterating $\varphi_t$, the points move away from one fixed points towards the other along geodesics, similarly to the behaviour of Hyperbolic Möbius transformations


Upper Half Plane
Poincare Disk

Let $\sigma = 1$, $\tau = -1$ and $\alpha = 1$.




Upper Half Plane
Poincare Disk
−5 −4 −3 −2 −1 0 1 2 3 4 5
t


Upper Half Plane
Poincare Disk
−5 −4 −3 −2 −1 0 1 2 3 4 5
t

# 12. On other Riemann Surfaces

**Definition 12.0.1 — Riemann Surface.**
A connected, Hausdorff, second countable topological space $S$ is a *Riemann Surface* if the exists an open covering $\{U_\alpha\}$ of $S$, continuous maps $\psi_\alpha : U_\alpha \mapsto \mathbb{C}$ such that $\psi_\alpha(U_\alpha)$ is an open subset of $\mathbb{C}$ and $\psi_\alpha : U_\alpha \mapsto \psi_\alpha(U_\alpha)$ is a homeomorphism and such that for every $\alpha, \beta$ with $U_\alpha \cap U_\beta \neq \emptyset$ the map,

$$\psi_\alpha \circ \psi_\beta^{-1} : \psi_\beta(U_\alpha \cap U_\beta) \mapsto \psi_\alpha(U_\alpha \cap U_\beta)$$

is holomorphic.

The family $\{U_\alpha, \psi_\alpha\}$ is called a *holomorphic atlas* for $S$ and $\psi_\alpha$ is a *holomorphic chart* of $S$ on $U_\alpha$. We defined holomorphic maps using the Cauchy Riemann Equations, and we now extend the notion of holomorphic to general maps of surfaces,

**Definition 12.0.2 — Holomorphic.**
A continuous map $f : S_1 \mapsto S_2$ between two Riemann surfaces $S_1, S_2$ is *holomorphic* if for every $p \in S_1$ there exists a holomorphic chart $(U, \psi)$ of $S_1$ with $p \in U$ and a holomorphic chart $(V, \eta)$ of $S_2$ with $f(p) \in V$ such that the function

$$\eta \circ f \circ \psi^{-1} : \psi(U \cap f^{-1}(V)) \mapsto \eta(V)$$

is holomorphic.

Before we discuss semi groups in the unit disk in detail, we briefly discuss other Riemann surfaces to justify this investigation. The following theorem allows us to reduce the space of Riemann surfaces down to 3 surfaces unique up to biholomorphism.

**Theorem 12.0.1 — Uniformization Theorem (1).**
Every simply connected Riemann surface is biholomorphic to either the unit disk $\mathbb{D}$, or the complex plane $\mathbb{C}$, or the Riemann sphere $\mathbb{C}_\infty$.

We can give a complete classification of semi groups in $\mathbb{C}$ and $\mathbb{C}_\infty$.

**Theorem 12.0.2 — Semigroups in $\mathbb{C}$ (4).**
Let $(\varphi_t)$ be a non trivial continuous semigroup in $\mathbb{C}$. Then there exists an affine transformation $T$ in $\mathbb{C}$ such that either,

1. $T \circ \varphi_t \circ T^{-1}(z) = z + it$, or
2. $T \circ \varphi_t \circ T^{-1}(z) = e^{at}z$, for some non zero $a \in \mathbb{C}$

In particular, every continuous semigroup in $\mathbb{C}$ is a continuous group.

**Theorem 12.0.3 — Semigroups in $\mathbb{C}_\infty$ (4).**
Let $(\varphi_t)$ be a non trivial semigroup in $\mathbb{C}_\infty$. Then there exists a Mobius transformation $T$ such that either,

1. $T \circ \varphi_t \circ T^{-1}(z) = z + it$, or
2. $T \circ \varphi_t \circ T^{-1}(z) = e^{at}z$, for some non zero $a \in \mathbb{C}$

In particular, every continuous semigroup in $\mathbb{C}_\infty$ is a continuous group.

Thus the unit disk is the most interesting domain in which to study semi groups, in particular semi groups which are not groups since we have a complete classification of groups in the unit disk.

# VI Models

# 13. Holomorphic Models

This chapter follows Chapter 9 of [4]. The idea behind models is to model the dynamic behaviour of a semigroup in $\mathbb{D}$ using a group on a Riemann surface. A candidate for models is the property of semi conjugation,

**Definition 13.0.1** Let $(\phi_t)$ be a semigroup in $\mathbb{D}$. Let $\Omega$ be a Riemann Surface and $(\psi_t)$ a semigroup in $\Omega$. $(\phi_t)$ is semi conjugated to $(\psi_t)$ if there exists a holomorphic map $g : \mathbb{D} \mapsto \Omega$ such that for all $t \geq 0$,

$$\begin{array}{ccc} \mathbb{D} & \xrightarrow{\phi_t} & \mathbb{D} \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g} \\ \Omega & \xrightarrow{\psi_t} & \Omega \end{array}$$

is a commutative diagram.

To motivate the definition of a model, lets investigate semi conjugation for two similar semi groups. Let $\psi_t(z) = z + it$, $(\psi_t)$ is a group of automorphisms of $\mathbb{C}$. Let $C_1 : \mathbb{D} \mapsto H = \{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0\}$ be the holomorphic map defined by $C_1(z) = \frac{1+z}{1-z}$. Thus we define the following two semi-groups in $\mathbb{D}$, $\phi_t, \tilde{\phi}_t : \mathbb{D} \mapsto \mathbb{D}$ defined by $\phi_t = C_1^{-1} \circ \psi_t \circ C_1$ and $\tilde{\phi}_t = C_1^{-1}(-i\psi_t(iC_1(z)))$. By construction $\phi_t$ is a non elliptic group in $\mathbb{D}$ and $\tilde{\phi}_t$ is a non elliptic semigroup in $\mathbb{D}$ that is not a group. Both semi groups are conjugated to $\psi_t$ using the maps $z \mapsto C_1(z)$ and $z \mapsto iC_1(z)$ respectively. Therefore $(\phi_t)$ being semi conjugated to $(\psi_t)$ is not enough to capture the dynamical behaviour of the semigroup.

**Definition 13.0.2 — Semi-Model.**
Let $(\phi_t)$ be a semigroup in $\mathbb{D}$. Let $\Omega$ be a Riemann Surface, $h : \mathbb{D} \mapsto \Omega$ holomorphic and $(\psi_t)$ a continuous group in $\Omega$ such that $(\phi_t)$ is semi conjugated to $(\psi_t)$ through $h$, namely the following diagram is commutative,

$$\begin{array}{ccc} \mathbb{D} & \xrightarrow{\phi_t} & \mathbb{D} \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h} \\ \Omega & \xrightarrow{\psi_t} & \Omega \end{array} \tag{13.1}$$

along with the condition

$$\bigcup_{t \leq 0} \psi_t \circ h(\mathbb{D}) = \Omega \tag{13.2}$$

Then the triple $(\Omega, h, \psi_t)$ is a semi-model for $(\phi_t)$.

If $h$ is injective, then $(\Omega, h, \psi_t)$ is a *model* for $(\phi_t)$. Firstly we prove some simple properties of (semi)models,

**Proposition 13.0.1** Let $(\phi_t)$ be a semigroup in $\mathbb{D}$ and $(\Omega, h, \psi_t)$ a semi model for $(\phi_t)$. Then

1. $\psi_t \circ h(\mathbb{D}) \subset h(\mathbb{D})$ for all $t \geq 0$. In particular, if $(\Omega, h, \psi_t)$ is a model, then $\phi_t = h^{-1} \circ \psi_t \circ h$ on $h(\mathbb{D})$.
2. If $(\phi_t)$ is elliptic, $(\psi_t)$ has a common fixed point.

**Proof**
1) Using the commutative diagram 1,

$$\psi_t \circ h(\mathbb{D}) = h \circ \phi_t(\mathbb{D}) \subset h(\mathbb{D})$$

If $(\Omega, h, \psi_t)$ is a model, $h$ is injective and thus bijective on its image, $h(\mathbb{D})$. Thus from the commutative diagram (13.1) it follows that $\phi_t = h^{-1} \circ \psi_t \circ h$ on $h(\mathbb{D})$.

2) If $(\phi_t)$ is elliptic the there exists $z_0 \in \mathbb{D}$ such that $\phi_t(z_0) = z_0$, thus

$$\psi_t \circ h(z_0) = h \circ \phi_t(z_0) = h(z_0)$$

Thus $h(z_0)$ is a fixed point of $(\psi_t)$

Let $\phi_t(z) = C_1^{-1}(e^t C_1(z))$. $(\phi_t)$ is a hyperbolic group in $\mathbb{D}$ and it has a model $(H, C_1, z \mapsto e^t z)$. We can construct another model for $(\phi_t)$ using $h : \mathbb{D} \mapsto S_\pi = \{z \in \mathbb{C} \mid 0 < \mathrm{Re}(z) < \pi\}$ defined by

$$h(z) = i \ln C_1(z) + \frac{\pi}{2}$$

where ln is the principal branch of the logarithm. Then $(S_\pi, h, z \mapsto z + it)$ is a model for $(\phi_t)$. Therefore a semigroup may have many different (semi) models, so to establish a unique model for each semigroup, we need a way of mapping between (semi) models.

**Definition 13.0.3** Let $(\phi_t)$ be a semigroup in $\mathbb{D}$. Let $(\Omega, h, \psi_t)$ and $(\tilde{\Omega}, \tilde{h}, \tilde{\psi}_t)$ be two semi models for $(\phi_t)$. A morphism of holomorphic semi models $\hat{\eta} : (\Omega, h, \psi_t) \mapsto (\tilde{\Omega}, \tilde{h}, \tilde{\psi}_t)$ is a holomorphic map $\eta : \Omega \mapsto \tilde{\Omega}$ such that the following are commutative diagrams.

$$\begin{array}{ccc} \mathbb{D} & \xrightarrow{h} & \Omega \\ \downarrow{\scriptstyle \tilde{h}} & \swarrow{\scriptstyle \eta} & \\ \tilde{\Omega} & & \end{array} \qquad \begin{array}{ccc} \Omega & \xrightarrow{\psi_t} & \Omega \\ \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \eta} \\ \tilde{\Omega} & \xrightarrow{\tilde{\psi}_t} & \tilde{\Omega} \end{array} \tag{13.3}$$

**Example 13.1 — Identity Morphism.**
Let $(\phi_t)$ be a semigroup in $\mathbb{D}$. Let $(\Omega, h, \psi_t)$ be a semi model for $(\phi_t)$. Let $id_\Omega : \Omega \mapsto \Omega$ be the

identity map, then

$$id_\Omega \circ h = h \qquad id_\Omega \circ \psi_t = \psi_t = \psi_t \circ id_\Omega$$

Therefore $\hat{id}_\Omega : (\Omega, h, \psi_t) \mapsto (\Omega, h, \psi_t)$ is a morphism of semi models.

**Example 13.2 — Composition of Morphisms.**
Let $(\phi_t)$ be a semigroup in $\mathbb{D}$. Let $(\Omega^1, h^1, \psi_t^1)$, $(\Omega^2, h^2, \psi_t^2)$, $(\Omega^3, h^3, \psi_t^3)$ be semi models for $(\phi_t)$. Suppose $\hat{\eta}_1 : (\Omega^1, h^1, \psi_t^1) \mapsto (\Omega^2, h^2, \psi_t^2)$ and $\hat{\eta}_2 : (\Omega^2, h^2, \psi_t^2) \mapsto (\Omega^3, h^3, \psi_t^3)$ are morphisms of semi models, then we have a natural composition of semi model morphisms, $\hat{\eta} : (\Omega^1, h^1, \psi_t^1) \mapsto (\Omega^3, h^3, \psi_t^3)$ defined by,

$$\hat{\eta} = \hat{\eta}_2 \circ \hat{\eta}_1 = \widehat{\eta_2 \circ \eta_1}$$

The definition of a semi model morphism gives us a lot of structure, which leads to the following two propositions,

**Proposition 13.0.2** Let $(\phi_t)$ be a semigroup in $\mathbb{D}$. Let $(\Omega, h, \psi_t)$ and $(\tilde{\Omega}, \tilde{h}, \tilde{\psi}_t)$ be two semi models for $(\phi_t)$. If $\hat{\eta} : (\Omega, h, \psi_t) \mapsto (\tilde{\Omega}, \tilde{h}, \tilde{\psi}_t)$ is a morphism of semi models, then $\eta : \Omega \mapsto \tilde{\Omega}$ is surjective.

**Proof**
Using the commutative diagrams for semi models and morphisms,

$$\begin{aligned} \tilde{\Omega} &= \bigcup_{t \le 0} \tilde{\psi}_t \circ \tilde{h}(\mathbb{D}) && (13.2) \\ &= \bigcup_{t \le 0} \tilde{\psi}_t \circ \eta \circ h(\mathbb{D}) && (13.3\ \mathrm{L}) \\ &= \bigcup_{t \le 0} \eta \circ \psi_t \circ h(\mathbb{D}) && (13.3\ \mathrm{R}) \\ &= \eta(\Omega) && (13.2) \end{aligned}$$

The last equality follows from the continuity of $\eta$, as $\eta$ is holomorphic.

A surprising result of the definition for semi models morphisms is the uniqueness of the morphisms,

**Proposition 13.0.3** Let $(\phi_t)$ be a semigroup in $\mathbb{D}$. Let $(\Omega, h, \psi_t)$ and $(\tilde{\Omega}, \tilde{h}, \tilde{\psi}_t)$ be two semi models for $(\phi_t)$. Suppose $\hat{\eta}, \hat{\mu} : (\Omega, h, \psi_t) \mapsto (\tilde{\Omega}, \tilde{h}, \tilde{\psi}_t)$ are two morphisms of semi models, then $\hat{\eta} = \hat{\mu}$.

**Proof**
Equivalently, it suffices to show $\eta(z) = \mu(z)$ for all $z \in \Omega$. The idea is to use the first commutative diagram of the morphism to swap $\eta$ and $\mu$, more explicitly $\eta \circ h = \tilde{h} = \mu \circ h$.

Let $z \in \Omega$, by the union condition, there exists $t \leq 0$ and $w \in \mathbb{D}$ such that $z = \psi_t \circ h(w)$,

$$\begin{aligned}\eta(z) &= \eta \circ \psi_t \circ h(w) \\ &= \tilde{\psi}_t \circ \eta \circ h(w) \\ &= \tilde{\psi}_t \circ \tilde{h}(w) \\ &= \tilde{\psi}_t \circ \mu \circ h(w) \\ &= \mu \circ \psi_t \circ h(w) \\ &= \mu(z)\end{aligned}$$

Similarly to groups we can define a more restricted morphism,

**Definition 13.0.4 — Isomorphism.**
Let $(\phi_t)$ be a semigroup in $\mathbb{D}$. Let $(\Omega, h, \psi_t)$ and $(\tilde{\Omega}, \tilde{h}, \tilde{\psi}_t)$ be two semi models for $(\phi_t)$. If $\hat{\eta} : (\Omega, h, \psi_t) \mapsto \tilde{\Omega}, \tilde{h}, \tilde{\psi}_t)$ is a morphism of semi models and there exists a morphism of semigroups $\hat{mu} : (\tilde{\Omega}, \tilde{h}, \tilde{\psi}_t) \mapsto (\Omega, h, \psi_t)$ such that $\hat{\mu} \circ \hat{\eta} = id_\Omega$ and $\hat{\eta} \circ \hat{\mu} = id_{\tilde{\Omega}}$ then $\hat{\eta}$ is an isomorphism of semi models.

The previous proposition allows us to give some equivalent definitions of semi model isomorphism.

**Theorem 13.0.4** Let $(\phi_t)$ be a semigroup in $\mathbb{D}$. Let $(\Omega, h, \psi_t)$ and $(\tilde{\Omega}, \tilde{h}, \tilde{\psi}_t)$ be two semi models for $(\phi_t)$. Let $\hat{\eta} : (\Omega, h, \psi_t) \mapsto \tilde{\Omega}, \tilde{h}, \tilde{\psi}_t)$ be a morphism of semi models. Then the following are equivalent,

1. $\eta$ is a biholomorphism.
2. $\hat{\eta}$ is an isomorphism of semi models
3. There exists a morphism of semi models $\mu : (\tilde{\Omega}, \tilde{h}, \tilde{\psi}_t) \mapsto (\Omega, h, \psi_t)$

**Proof**
Since $\eta$ has a holomorphic inverse, this is the morphism in the other direction and thus $(1) \rightarrow (2)$. By definition $(2) \rightarrow (3)$. Suppose $(3)$ holds, then $\hat{\mu} \circ \hat{\eta}$ is an endomorphism of $(\Omega, h, \psi_t)$ and thus $\mu \circ \eta = id_\Omega$. Therefore $\eta$ is biholomorphic with holomorphic inverse $\mu$, $(3) \rightarrow (1)$.

The final proposition before we prove the main result gives us a morphism from a model to any semi model,

**Proposition 13.0.5** Let $(\phi_t)$ be a semigroup in $\mathbb{D}$. Let $(\Omega, h, \psi_t)$ be a model for $(\phi_t)$. Suppose $(\tilde{\Omega}, \tilde{h}, \tilde{\psi}_t)$ is a semi model for $(\phi_t)$, then there exists a unique morphism of semi models $\hat{\eta} : (\Omega, h, \psi_t) \mapsto \tilde{\Omega}, \tilde{h}, \tilde{\psi}_t)$.

**Proof**
Uniqueness follows from Proposition 16.0.3. Let $\Omega_t = \psi_{-t} \circ h(\mathbb{D})$ and $\eta_t : \Omega_t \mapsto \tilde{\Omega}$ be defined

by $\eta_t = \tilde{\psi}_{-t} \circ \tilde{h} \circ h^{-1}|_{h(\mathbb{D})} \circ \psi_t$.

Let $0 \le t \le s$ and $z \in \Omega_t \cap \Omega_s$, then there exists $w \in \mathbb{D}$ such that $z = \phi_{-t} \circ h(w) = \phi_{-s} \circ h(w)$

$$\psi_s \circ h(w) = \psi_t \circ h(w)$$
$$h \circ \phi_s(w) = h \circ \phi_t(w)$$

Since $h$ is injective,

$$\phi_s(w) = \phi_t(w)$$
$$\tilde{h} \circ \phi_s(w) = \tilde{h} \circ \phi_t(w)$$
$$\tilde{\psi}_s \circ \tilde{h}(w) = \tilde{\psi}_t \circ \tilde{h}(w)$$

Thus $\tilde{\psi}_{-s} \circ \tilde{h}(w) = \tilde{\psi}_{-t} \circ \tilde{h}(w)$ which be useful next. Now we show that $\eta_t(z) = \eta_s(z)$,

$$\begin{aligned}
\eta_t(z) &= \tilde{\psi}_{-t} \circ \tilde{h} \circ h^{-1}|_{h(\mathbb{D})} \circ \psi_t(z) \\
&= \tilde{\psi}_{-t} \circ \tilde{h} \circ h^{-1}|_{h(\mathbb{D})} \circ \psi_t \circ \phi_{-t} \circ h(w) \\
&= \tilde{\psi}_{-t} \circ \tilde{h}(w) \\
&= \tilde{\psi}_{-s} \circ \tilde{h}(w) \\
&= \tilde{\psi}_{-s} \circ \tilde{h} \circ h^{-1}|_{h(\mathbb{D})} \circ \psi_s \circ \phi_{-s} \circ h(w) \\
&= \tilde{\psi}_{-s} \circ \tilde{h} \circ h^{-1}|_{h(\mathbb{D})} \circ \psi_s(z) \\
&= \eta_s(z)
\end{aligned}$$

By definition, $\cup_{t \le 0} \Omega_t = \Omega$. Let $\eta : \Omega \mapsto \tilde{\Omega}$ be defined by $\eta(z) = \eta_t(z)$ for some $t \ge 0$ such that $z \in \Omega_t$. Therefore it suffices to show the two morphism diagrams commute.

Let $z \in \Omega$. Let $t \ge 0$ and $w \in \mathbb{D}$ such that $z = \psi_{-t} \circ h(w)$.

$$\begin{aligned}
\eta \circ h(w) &= \tilde{\psi}_{-t} \circ \tilde{h} \circ h^{-1}|_{h(\mathbb{D})} \circ \psi_t \circ h(w) \\
&= \tilde{\psi}_{-t} \circ \tilde{h} \circ h^{-1}|_{h(\mathbb{D})} \circ h \circ \phi_t(w) \\
&= \tilde{\psi}_{-t} \circ \tilde{h} \circ \phi_t(w) \\
&= \tilde{\psi}_{-t} \circ \tilde{\psi}_t \circ \tilde{h}(w) \\
&= \tilde{h}(w)
\end{aligned}$$

Similarly for the second diagram,

$$\begin{aligned}
\tilde{\psi}_t \circ \eta(z) &= \tilde{\psi}_t \circ \tilde{\psi}_{-t} \circ \tilde{h} \circ h^{-1}|_{h(\mathbb{D})} \circ \psi_t(z) \\
&= \tilde{\psi}_{-t} \circ \tilde{\psi}_t \circ \tilde{h} \circ h^{-1}|_{h(\mathbb{D})} \circ \psi_t(z) \\
&= \tilde{\psi}_{-t} \circ \tilde{h} \circ \phi_t \circ h^{-1}|_{h(\mathbb{D})} \circ \psi_t(z) \\
&= \tilde{\psi}_{-t} \circ \tilde{h} \circ h^{-1}|_{h(\mathbb{D})} \psi_t \circ \psi_t(z) \\
&= \eta \circ \psi(z)
\end{aligned}$$

**Corollary 13.0.6** Models of a semigroups $(\phi_t)$ are unique up to semi model isomorphism

The previous proposition gives a morphism in both directions, so by the characterisation of semi model isomorphism, the models are isomorphic.

Thus if we consider a partial ordering on the semi models of a semigroup defined by $A \geq B$ iff there exists a morphism of semimodels $\hat{\eta} : A \mapsto B$, then a model, if it exists, is the unique maximal element of the semi models of the semigroup.

We are now ready to prove the main result of this section

**Theorem 13.0.7** Let $(\phi_t)$ be a semigroup in $\mathbb{D}$. Then there exists a unique (up to semimodel isomorphism) model $(\Omega, h, \psi_t)$ for $(\phi_t)$.

**Proof**
Uniqueness follows from the previous corollary, so we proceed with existence. The outline of the proof is as follows,

1. Construct a topological space $\Omega$ and show it is a Riemann Surface
2. Construct an intertwining map $h : \mathbb{D} \mapsto \Omega$ and show it is injective
3. Construct an algebraic group $(\psi_n)$ in $\Omega$
4. Show that $(\Omega, h, \psi_t)$ forms a model for $(\phi_n)$.

1) Firstly we construct the topological space $\Omega$. Let $[0, \infty)$ have the discrete topology. Define a relation on $\mathbb{D} \times [0, \infty)$ as $(z,t) \sim (w,s)$ iff there exists $u \in [0, \infty)$ such that $u \geq \max\{t,s\}$ and $\phi_{u-t}(z) = \phi_{u-s}(w)$. $\sim$ is an equivalence relation, each of the properties follow from $=$ being an equivalence relation. Let $\Omega$ be the quotient space $\mathbb{D} \times [0, \infty)/\sim$ and $\pi$ be the identification map.

To show $\Omega$ is a Riemann Surface, we need a holomorphic atlas $\{(h_t, \Omega_t)\}$ and we need to show that $\Omega$ is second countable and path connected. Fix $t \geq 0$ and define $h_t : \mathbb{D} \mapsto \Omega$ by $h_t(z) = \pi((z,t))$. $h_t$ is injective as $\phi_t$ is injective. $h_t$ is continuous by the continuity of $\pi$. To show $h_t$ is an open map let $U \subset \mathbb{D}$ be open. A set in the quotient space is open if its preimage is open, so we need to check if $\pi^{-1}(\pi((U,t)))$ is open in $\mathbb{D} \times [0, \infty)$. As this is a product space it suffices to check that the projections are open. The projection onto $[0, \infty)$ is clearly open as $[0, \infty)$ has the discrete topology. For the projection onto $\mathbb{D}$, note $\{z \in \mathbb{D} \mid (z,t) \in [(U,t)]\} = U$ as $\phi_n$ is injective. Thus the projection $\pi^{-1}(\pi((U,t)))$ onto $\mathbb{D}$ is the union of open sets, which is open. Therefore $h_t|_{h(\mathbb{D})}$ is a homeomorphism.

Define $\Omega_t = h_t(\mathbb{D})$. Fix $s \geq 0$. Clearly $(\phi_{t-s}, t) \sim (z, s)$ for all $t \geq s$. Thus,

$$h_s = h_t \circ \phi_{t-s}$$

In particular,

$$\Omega_s = h_s(\mathbb{D}) = h_t \circ \phi_{t-s}(\mathbb{D}) \subset h_t(\mathbb{D}) = \Omega_t$$

Therefore $\Omega = \cup_{t \in \mathbb{N}} \Omega_t$, and is therefore second countable and path connected. It follows that $\{(h_t, \Omega_t)\}$ is a holomorphic atlas for $\Omega$ and $\Omega$ is a Riemann Surface.

2) Let $h = h_0 : \mathbb{D} \mapsto \Omega$. $h$ is injective by construction.

3) Define $\psi_s : \Omega \mapsto \Omega$ by

$$\phi_s(z) = h_{t-s} \circ h_t^{-1} \qquad z \in \Omega_t$$

Fix $s \geq 0$ then for all $t \geq s$,

$$\begin{aligned} h_{t-s} \circ h^{-1}|_{\Omega_s} &= h_{t-s} \circ h_t^{-1} \circ h_s \circ h_s^{-1} \\ &= h_{t-s} \circ \phi_{t-s} \circ h_s^{-1} \\ &= h \circ h_s^{-1} \end{aligned}$$

Thus the map $\psi_s$ is well defined. Since $h$ is injective, $\psi$ is injective.

Let $w \in \Omega$ and fix $s \geq 0$. Let $t \geq s$ such that $w \in \Omega_{t-s}$ then there exists $u \in \mathbb{D}$ such that $w = h_{t-s}(u)$. Let $z = h_t(u)$.

$$\begin{aligned} w &= h_{t-s}(u) \\ &= h_{t-s} \circ h_t^{-1}(z) \\ &= \psi_s(z) \end{aligned}$$

Therefore $\psi_s$ is injective. Finally we need to show that $(\psi_n)$ forms an algebraic group. It suffices to show that $\psi_t \circ \psi_s = \psi_{t+s}$. Let $t, s \geq 0$ and $z \in \Omega$. There exists $u \geq t+s$ such that $s \in \Omega_u$. Then $z = h_u(w)$ for some $w \in \mathbb{D}$.

$$\begin{aligned} \psi_t \circ \psi_s(z) &= h_{u-t} \circ h_u^{-1} \circ h_{u-s} \circ h_u^{-1}(z) \\ &= h_{u-t} \circ h_u^{-1} \circ h_{u-s} \circ h_u^{-1} \circ h_u(w) \\ &= h_{u-t} \circ h_u^{-1} \circ h_{u-s}(w) \\ &= h_{u-t} \circ h_u^{-1} \circ h_u \circ \phi_s(w) \\ &= h_{u-t} \circ \phi_s(w) \\ &= h_u \circ \phi_t \circ \phi_s(w) \\ &= h_u \circ \phi_{t+s}(w) \\ &= h_{u-(t+s)}(w) \\ &= h_{u-(t+s)} \circ h_u^{-1}(z) \\ &= \psi_{t+s}(z) \end{aligned}$$

The continuity of $(\psi_n)$ follows from the continuity of $(\phi_n)$ (see [4]). Therefore $(\psi_n)$ is an algebraic group in $\Omega$.

4) Let $t \geq 0$, since $h(\mathbb{D}) \subset \Omega_t$,

$$\psi_t \circ h = h \circ h_t^{-1} \circ h = h \circ \phi_t$$

Since $\psi_{-t} = h_t \circ h^{-1}$ on $h(\mathbb{D})$,

$$\bigcup_{t \geq 0} \psi_{-t} \circ h(\mathbb{D}) = \bigcup_{t \geq 0} h_t(\mathbb{D}) = \Omega$$

Therefore $(\Omega, h, \psi_n)$ is a model for $(\phi_n)$.

We can extend the Denjoy Wolff theorem to continuous semigroups [4], and use this to classify semigroups similarly to holomorphic self maps,

**Theorem 13.0.8** Let $(\phi_t)$ be a non trivial semi groups in $\mathbb{D}$. Then, all iterates different from the identity have the same Denjoy-Wolff point $\tau \in \overline{\mathbb{D}}$.

**Theorem 13.0.9** Let $(\phi_n)$ be a semigroup in $\mathbb{D}$. Then,

1. $(\phi_n)$ is the trivial semigroup iff $(\phi_n)$ has a holomorphic model $(\mathbb{D}, id_{\mathbb{D}}, z \mapsto z)$
2. $(\phi_n)$ is a group of elliptic automorphisms iff $(\phi_n)$ has a holomorphic model $(\mathbb{D}, h, z \mapsto e^{-i\theta} z)$, $\theta \in \mathbb{R} - \{0\}$
3. $(\phi_n)$ is elliptic, not a group iff $(\phi_n)$ has a holomorphic model $(\mathbb{C}, h, z \mapsto e^{-\lambda t} z)$, $\lambda \in \mathbb{C}$ such that $\operatorname{Re} \lambda > 0$
4. $(\phi_n)$ is hyperbolic iff $(\phi_n)$ has a holomorphic model $(\mathbb{S}_{\frac{\pi}{\lambda}}, h, z \mapsto z + it)$, $\lambda > 0$
5. $(\phi_n)$ is parabolic of positive hyperbolic step iff $(\phi_n)$ has a holomorphic model either of the form $(\mathbb{H}, h, z \mapsto z + it)$ or $(\mathbb{H}^-, h, z \mapsto z + it)$.
6. $(\phi_n)$ is parabolic of zero hyperbolic step iff $(\phi_n)$ has a holomorphic model $(\mathbb{C}, h, z \mapsto z + it)$

# 14. Conclusion

In this report we have studied the geometric properties of the Poincaré disk and Upper half plane with the hyperbolic metric. We studied the dynamics of iterating Möbius transformations and characterised the different types. We introduced algebraic semigroups and models, one of the tools for characterising semigroups. Further research into this area could include studying other tools for characterisation such as infinitesimal generators which has applications to differential equations. Additionally the reference for this report ([4]) took an analytic approach to defining semigroups, a more abstract approach using using tangent bundles of intervals would allow the use of more powerful mathematical techniques as these spaces have been thoroughly studied, see [3].

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